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Curvature of multiply warped products

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Abstract

In this paper, we study Ricci-flat and Einstein–Lorentzian multiply warped products. We also consider the case of having constant scalar curvatures for this class of warped products. Finally, after we introduce a new class of space–times called as generalized Kasner space–times, we apply our results to this kind of space–times as well as other relativistic space–times, i.e., Reissner–Nordström, Kasner space–times, Bañados–Teitelboim–Zanelli and de Sitter black hole solutions.

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1. Introduction

The concept of warped products was first introduced by Bishop and O'Neill (see [17]) to construct examples of Riemannian manifolds with negative curvature. In Riemannian geometry, warped product manifolds and their generic forms have been used to construct new

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examples with interesting curvature properties since then (see [16,17,21,24,29,37,38,49–51,54,57]). In Lorentzian geometry, it was first noticed that some well known solutions to Einstein's field equations can be expressed in terms of warped products in [12] and after that Lorentzian warped products have been used to obtain more solutions to Einstein's field equations (see [12,13,16,17,41,56,62]). Moreover, geometric properties such as geodesic structure or curvature of Lorentzian warped products have been studied by many authors because of their relativistic applications (see [2–5,10,11,14,15,19,20,25,27,29–31,33,34,42,47,52,53,60,64–68,71,72]).

We recall the definition of a warped product of two pseudo-Riemannian manifolds (B, g_B) and (F, g_F) with a smooth function $b : B \to (0, \infty)$ (see also [13,62]). Suppose that (B, g_B) and (F, g_F) are pseudo-Riemannian manifolds and also suppose that $b : B \to (0, \infty)$ is a smooth function. Then the (singly) warped product, $B \times_b F$ is the product manifold $B \times F$ equipped with the metric tensor $g = g_B \oplus b^2 g_F$ defined by

$$g = \pi^*(g_B) \oplus (b \circ \pi)^2 \sigma^*(g_F),$$

where $\pi : B \times F \to B$ and $\sigma : B \times F \to F$ are the usual projection maps and $*$ denotes the pull-back operator on tensors. Here, (B, g_F) is called as the base manifold and (F, g_F) is called as the fiber manifold and also b is called as the warping function.

Generalized Robertson–Walker space–time models (see [2,11,33,65,67,68]) and standard static space–time models (see [3–5,52,53]) that are two well known solutions to Einstein's field equations can be expressed as Lorentzian warped products. Clearly, the former is a natural generalization of Robertson–Walker space–time and the latter is a generalization of Einstein static universe. One way to generalize warped products is to consider the case of multi fibers to obtain more general space–time models (see examples given in Section 2) and in this case the corresponding product is so called multiply warped product. In [72], covariant derivative formulas for multiply warped products are given and the geodesic equation for these spaces are also considered. The causal structure, Cauchy surfaces and global hyperbolicity of multiply Lorentzian warped products are also studied. Moreover, necessary and sufficient conditions are obtained for null, time-like and space-like geodesic completeness of Lorentzian multiply products and also geodesic completeness of Riemannian multiply warped products. In [19,20], the author studies manifolds with C^0 -metrics and properties of Lorentzian multiply warped products and then he shows a representation of the interior Schwarzschild space–time as a multiply warped product space–time with certain warping functions. He also gives the Ricci curvature in terms of b_1, b_2 for a multiply warped product of the form $M = (0, 2m) \times_{b_1} \mathbb{R}^1 \times_{b_2} \mathbb{S}^2$. In [42], physical properties (2 + 1) charged Bañados–Teitelboim–Zanelli (BTZ) black holes and (2 + 1) charged de Sitter (dS) black holes are studied by expressing these metrics as multiply warped product space–times, more explicitly, Ricci and Einstein tensors are obtained inside the event horizons (see also [9]). In [66], the existence, multiplicity and causal character of geodesics joining two points of a wide class of non-static Lorentz manifolds such as intermediate Reissner–Nordström or inner Schwarzschild and generalized Robertson–Walker space–times are studied. In [34], geodesic connectedness and also causal geodesic connectedness of multi-warped space–times are studied by using the method of Brouwer's topological degree for the solution of functional equations. There are also different types of warped products such as a kind of warped product with two warping functions acting symmetrically on the fiber and base manifolds, called as a doubly warped product (see [71]) or another kind of warped product called as a twisted product when the warping function defined on the product of the base and fiber manifolds (see [32]). Moreover, Easley studied *local existence warped product structures* and also defined and considered another form of a warped product in his thesis (see [28]).

In this paper, we answer some questions about the existence of nontrivial warping functions for which the multiply warped product is Einstein or has a constant scalar curvature. This problem was considered especially for Einstein Riemannian warped products with compact base and some partial answers were also provided (see [38,49–51]). In [50], it is proved that an Einstein Riemannian warped product with a non-positive scalar curvature and compact base is just a trivial Riemannian product. Constant scalar curvature of warped products was studied in [22,24,29,30] when the base is compact and of generalized Robertson–Walker space–times in [29]. Furthermore, partial results for warped products with non-compact base were obtained in [7,18]. The physical motivation of existence of a positive scalar curvature comes from the positive mass problem. More explicitly, in general relativity the positive mass problem is closely related to the existence of a positive scalar curvature (see [75]). As a more general related reference, one can consider [48] to see a survey on scalar curvature of Riemannian manifolds. The problem of existence of a warping function which makes the warped product Einstein was already studied for special cases such as generalized Robertson–Walker space–times and a table given the different cases of Einstein generalized Robertson–Walker when the Ricci tensor of the fiber is Einstein in [2] (see also references therein). Einstein–Ricci tensor and constant scalar curvature of standard static space–times with perfect fluid were already considered in [52,60]. Moreover, in [53], the conformal tensor on standard static space–times with perfect fluid is studied and it is shown that a standard static space–time with perfect fluid is conformally flat if and only if its fiber is Einstein and hence of constant curvature. In [25], this problem is considered for arbitrary standard static space–times, more explicitly, an essential investigation of conditions for the fiber and warping function for a standard static space–time (not necessarily with perfect fluid) is carried out so that there exists no nontrivial function on the fiber guaranteeing that the standard static space–time is Einstein. Duggal studied the scalar curvature of four-dimensional triple Lorentzian products of the form $L \times B \times_f F$ and obtained explicit solutions for the warping function f to have a constant scalar curvature for this class of products (see [27]). Moreover, in the present paper, we introduce an original form to generalize Kasner space–times and then we obtain necessary and sufficient conditions as well as explicit solutions, for some special cases, for a generalized Kasner space–time to be Einstein or to have constant scalar curvature. Besides than the form mentioned here, there are also other generalizations in the literature (see [43,55]). In [43], an extension for Kasner space–times is introduced in the view of generalizing five-dimensional Randall–Sundrum model to higher dimensions and in [55], another multi-dimensional generalization of Kasner metric is described and essential solutions are also obtained for this class of extension. One can also consider [23,36,44,45,58,63,73] for recent applications of Kasner metrics and its generalizations.

We organize the paper as follows. In Section 2, we give several basic geometric facts related to the concept of curvatures (see [70,72]). Moreover, we recall two well known examples of relativistic space–times which can be considered as generalized multiply Robertson–Walker space–times. In Section 3, we obtain two results in which, under several assumptions

Table 1

ζ	η	$\frac{\eta}{\zeta^2}$	λ	λ_{F_2}	p_1	p_2	Metric	φ
0	0	–	0	0	0	0	$-\mathrm{d}t^2 + g_{F_1} + g_{F_2}$	–
0	$\frac{3}{2}p_1^2 \neq 0$	–	0	0	$\neq 0$	$-\frac{1}{2}p_1$	$-\mathrm{d}t^2 + \varphi_0^{2p_1} g_{F_1} + \varphi_0^{-p_1} g_{F_2}$	$\varphi_0 = cte > 0$
0	$\frac{3}{2}p_1^2 \neq 0$	–	–	$\neq 0$	$\neq 0$	$-\frac{1}{2}p_1$	No metric	–
$\neq 0$	ζ^2	1	0	0	$\neq 0$	$0, -2p_1$	$-\mathrm{d}t^2 + \varphi^{2p_1} g_{F_1} + \varphi^{2p_2} g_{F_2}$	$(\varphi^\zeta; 0)$
$\neq 0$	ζ^2	1	$\neq 0$	λ	$\neq 0$	0	$-\mathrm{d}t^2 + \varphi^{2p_1} g_{F_1} + g_{F_2}$	$(\varphi^\zeta; \lambda)$
$\neq 0$	$\neq 0$	$\neq 1$	0	0	p_1	$\neq 0$	$-\mathrm{d}t^2 + \varphi_0^{2p_1} g_{F_1} + \varphi_0^{2p_2} g_{F_2}$	$\varphi_0 = cte > 0$
$\neq 0$	$\neq 0$	$\neq 1$	0	< 0	0	$\neq 0$	$-\mathrm{d}t^2 + g_{F_1} + \varphi^{2p_2} g_{F_2}$	$(\varphi^{\eta/\zeta}; 0)$
$\neq 0$	$\neq 0$	$\neq 1$	> 0	0	p_2	$\neq 0$	$-\mathrm{d}t^2 + \varphi^{2p_1} g_{F_1} + \varphi^{2p_1} g_{F_2}$	$(\varphi^\zeta; 3\lambda; *)$
$\neq 0$	$\neq 0$	$\neq 1$	$\neq 0$	$\neq 0$	p_1	$\neq 0$	No metric	–

Table 2

ζ	η	$\frac{\eta}{\zeta^2}$	τ_{F_2}	p_1	p_2	φ eq.
0	0	–	τ_{F_2}	0	0	$\tau = \tau_{F_2}$
0	$\frac{3}{2}p_1^2$	–	0	$\neq 0$	$-\frac{1}{2}p_1$	$\tau = \eta\frac{(\varphi')^2}{\varphi^2}$
0	$\frac{3}{2}p_1^2$	–	$\neq 0$	$\neq 0$	$-\frac{1}{2}p_1$	$\tau = \eta\frac{(\varphi')^2}{\varphi^2} + \frac{\tau_{F_2}}{\varphi^{2p_2}}$
$\zeta \neq 0$	ζ^2	1	0	$\neq 0$	0	$-2u'' = -\tau u; u = \varphi^\zeta$
$\zeta \neq 0$	ζ^2	1	0	$\neq 0$	$-2p_1$	$-2u'' = -\tau u; u = \varphi^\zeta$
$\zeta \neq 0$	ζ^2	1	$\neq 0$	$\neq 0$	0	$-2u'' = -(\tau - \tau_{F_2}); u = \varphi^\zeta$
$\zeta \neq 0$	ζ^2	1	$\neq 0$	$\neq 0$	$-2p_1$	$-2u'' = -\tau u + \tau_{F_2} u^{-1/3}; u = \varphi^\zeta$
$\zeta \neq 0$	$\eta \neq 0$	$\neq 1$	0	p_1	$\neq 0$	(5.1); $u = (\varphi^\zeta)^{(1+\eta/\zeta^2)/2}$
$\zeta \neq 0$	$\eta \neq 0$	$\neq 1, \frac{1}{3}$	$\neq 0$	p_1	$\neq 0$	(csc-K-II.c); $u = (\varphi^\zeta)^{(1+\eta/\zeta^2)/2}$
$\zeta \neq 0$	$\frac{\zeta^2}{3}$	$\frac{1}{3}$	$\neq 0$	$\frac{\zeta}{3}$	$\frac{\zeta}{3}$	$-3u'' = -\tau u + \tau_{F_2}; u = \varphi^{(2/3)\zeta}$

on the fibers and warping functions, multiply generalized Robertson–Walker space–times are Einstein or have constant scalar curvature. In Section 4, after we introduce generalized Kasner space–times, we state conditions for this class of space–times to be Einstein or to have constant scalar curvature. In Section 5, we give an explicit classification of four-dimensional multiply generalized Robertson–Walker space–times and four-dimensional generalized Kasner space–times which are Einstein. In the last section, we focus on BTZ $(2+1)$ -black hole solutions and classify (BTZ) black hole solutions given in Section 2 by using a more formal approach (see [8,9,42,59]) and then we also prove necessary and sufficient conditions for the lapse function of a BTZ $(2+1)$ black hole solution to have a constant scalar curvature or to be Einstein. Our main results are obtained in Sections 3–5, especially see Theorem 3.3, Propositions 4.3 and 4.11 as well as Tables 1–3.

Table 3

ζ	η	$\frac{\eta}{\zeta^2}$	λ	p_1	p_2	p_3	Metric	φ
0	0	–	0	0	0	0	$-\mathrm{d}t^2 + g_{F_1} + g_{F_2} + g_{F_3}$	–
0	$\neq 0$	–	0	p_1	p_2	p_3	$-\mathrm{d}t^2 + \varphi_0^{2p_1} g_{F_1} + \varphi_0^{2p_2} g_{F_2} + \varphi_0^{2p_3} g_{F_3}$	$\varphi_0 = cte > 0$
$\neq 0$	ζ^2	1	0	p_1	p_2	p_3	$-\mathrm{d}t^2 + \varphi^{2p_1} g_{F_1} + \varphi^{2p_2} g_{F_2} + \varphi^{2p_3} g_{F_3}$	$(\varphi^\zeta; 0)$
$\neq 0$	$\neq 0$	$\neq 1$	0	p_1	p_2	p_3	$-\mathrm{d}t^2 + \varphi_0^{2p_1} g_{F_1} + \varphi_0^{2p_2} g_{F_2} + \varphi_0^{2p_3} g_{F_3}$	$\varphi_0 = cte > 0$
$\neq 0$	$\neq 0$	$\neq 1$	> 0	p_1	p_1	p_1	$-\mathrm{d}t^2 + \varphi^{2p_1} g_{F_1} + \varphi^{2p_1} g_{F_2} + \varphi^{2p_1} g_{F_3}$	$(\varphi^\zeta; 3\lambda; *)$

2. Preliminaries

Throughout this work any manifold M is assumed to be connected, Hausdorff, paracompact and smooth. Moreover, I denotes for an open interval in $\mathbb{R}$ of the form $I = (t_1, t_2)$ where $-\infty \leq t_1 < t_2 \leq \infty$ and we will furnish I with a negative metric $-\mathrm{d}t^2$. A pseudo-Riemannian manifold (M, g) is a smooth manifold with a metric tensor g and a Lorentzian manifold (M, g) is a *pseudo-Riemannian* manifold with signature $(-, +, +, \ldots, +)$. Moreover, we use the definition and the sign convention for the *curvature* as in [13]. For an arbitrary n-dimensional pseudo-Riemannian manifold (M, g) and a smooth function $f : M \to \mathbb{R}$, we have that H^f and $\Delta(f)$ denote the *Hessian* (0, 2) tensor and the Laplace–Beltrami operator of f, respectively [62]. Here, we use the sign convention for the Laplacian in [62], i.e., defined by $\Delta = \mathrm{tr}_g(\mathrm{H})$ (see p. 86 of [62]) where H is the Hessian form (see p. 86 of [62]) and tr_g denotes for the trace, or equivalently, $\Delta = \mathrm{div}(\mathrm{grad})$, where div is the divergence and grad is the gradient (see p. 85 of [62]). Furthermore, we will frequently use the notation $\|\mathrm{grad}\, f\|^2 = g(\mathrm{grad}\, f, \mathrm{grad}\, f)$. When there is a possibility any misunderstanding, we will explicitly state the manifold or the metric for which the operator is considered.

We begin our discussion by giving the formal definition of a multiply warped product (see [72]).

Definition 2.1. Let (B, g_B) and (F_i, g_{F_i}) be *pseudo-Riemannian* manifolds and also let $b_i : B \to (0, \infty)$ be smooth functions for any $i \in \{1, 2, \ldots, m\}$. The *multiply warped product* is the *product manifold* $M = B \times F_1 \times F_2 \times \cdots \times F_m$ furnished with the metric tensor $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus \cdots \oplus b_m^2 g_{F_m}$ defined by

$$g = \pi^*(g_B) \oplus (b_1 \circ \pi)^2 \sigma_1^*(g_{F_1}) \oplus \cdots \oplus (b_m \circ \pi)^2 \sigma_m^*(g_{F_m}). \tag{2.1}$$

Each function $b_i : B \to (0, \infty)$ is called a *warping* function and also each manifold (F_i, g_{F_i}) is called a fiber manifold for any $i \in \{1, 2, \ldots, m\}$. The manifold (B, g_B) is the base manifold of the multiply warped product:

- If $m = 1$, then we obtain a *singly warped product*.
- If all $b_i \equiv 1$, then we have a (trivial) *product manifold*.
- If (B, g_B) and (F_i, g_{F_i}) are all *Riemannian* manifolds for any $i \in \{1, 2, \ldots, m\}$, then (M, g) is also a *Riemannian* manifold.
- The multiply warped product (M, g) is a *Lorentzian multiply warped product* if (F_i, g_{F_i}) are all *Riemannian* for any $i \in \{1, 2, \ldots, m\}$ and either (B, g_B) is *Lorentzian* or else (B, g_B) is a one-dimensional manifold with a *negative definite* metric $-\mathrm{d}t^2$.
- If B is an open connected interval I of the form $I = (t_1, t_2)$ equipped with the negative definite metric $g_B = -\mathrm{d}t^2$, where $-\infty \leq t_1 < t_2 \leq \infty$, and (F_i, g_{F_i}) is Riemannian for any $i \in \{1, 2, \ldots, m\}$, then the Lorentzian multiply warped product (M, g) is called a multiply generalized Robertson–Walker space–time or a multi-warped space–time. In particular, a multiply generalized Robertson–Walker space–time is called a generalized Reissner–Nordström space–time when $m = 2$.

We will state the covariant derivative formulas for *multiply warped products* (see [19,70,72]).

Proposition 2.2. *Let $M = B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a pseudo-Riemannian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$ also let $X, Y \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F_i)$, $W \in \mathfrak{L}(F_j)$. Then*

(1) $\nabla_X Y = \nabla^B_X Y$.
(2) $\nabla_X V = \nabla_V X = \frac{X(b_i)}{b_i} V$.
(3) $\nabla_V W = \begin{cases} 0 & if\ i \neq j, \\ \nabla^{F_i}_V W - \frac{g(V,W)}{b_i} \mathrm{grad}_B b_i & if\ i = j. \end{cases}$

One can compute the *gradient* and the *Laplace–Beltrami* operator on M in terms of the *gradient* and the *Laplace–Beltrami* operator on B and F_i , respectively. From now on, we assume that $\Delta = \Delta_M$ and $\mathrm{grad} = \mathrm{grad}_M$ to simplify the notation.

Proposition 2.3. *Let $M = B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a pseudo-Riemannian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$ and $\phi : B \to \mathbb{R}$ and $\psi_i : F_i \to \mathbb{R}$ be smooth functions for any $i \in \{1, \ldots, m\}$. Then*

(1) $\mathrm{grad}(\phi \circ \pi) = \mathrm{grad}_B \phi$.
(2) $\mathrm{grad}(\psi_i \circ \sigma_i) = \dfrac{\mathrm{grad}_{F_i} \psi_i}{b_i^2}$.
(3) $\Delta(\phi \circ \pi) = \Delta_B \phi + \displaystyle\sum_{i=1}^{m} s_i \dfrac{g_B(\mathrm{grad}_B \phi, \mathrm{grad}_B b_i)}{b_i}$.
(4) $\Delta(\psi_i \circ \sigma_i) = \dfrac{\Delta_{F_i} \psi_i}{b_i^2}$.

Now, we will state *Riemannian curvature* and *Ricci curvature* formulas from [70].

Proposition 2.4. *Let $M = B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a pseudo-Riemannian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$ also let $X, Y, Z \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F_i)$, $W \in \mathfrak{L}(F_j)$ and $U \in \mathfrak{L}(F_k)$. Then*

(1) $R(X, Y)Z = R_B(X, Y)Z$.
(2) $R(V, X)Y = -\frac{\mathrm{H}^{b_i}_B(X,Y)}{b_i} V$.
(3) $R(X, V)W = R(V, W)X = R(V, X)W = 0$ *if* $i \neq j$.
(4) $R(X, Y)V = 0$.
(5) $R(V, W)X = 0$ *if* $i = j$.
(6) $R(V, W)U = 0$ *if* $i = j$ *and* $i, j \neq k$.
(7) $R(U, V)W = -g(V, W)\frac{g_B(\mathrm{grad}_B b_i, \mathrm{grad}_B b_k)}{b_i b_k} U$ *if* $i = j$ *and* $i, j \neq k$.
(8) $R(X, V)W = \frac{g(V,W)}{b_i} \nabla^B_X(\mathrm{grad}_B b_i)$ *if* $i = j$.
(9) $R(V, W)U = R_{F_i}(V, W)U + \frac{\|\mathrm{grad}_B b_i\|_B^2}{b_i^2}(g(V, U)W - g(W, U)V)$ *if* $i, j = k$.

Proposition 2.5. *Let $M = B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a pseudo-Riemannian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$, also let $X, Y, Z \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F_i)$ and $W \in \mathfrak{L}(F_j)$. Then*

(1) $\mathrm{Ric}(X,Y) = \mathrm{Ric}_B(X,Y) - \sum_{i=1}^{m} \frac{s_i}{b_i} \mathrm{H}_B^{b_i}(X,Y)$.

(2) $\mathrm{Ric}(X,V) = 0$.

(3) $\mathrm{Ric}(V,W) = 0$ *if* $i \neq j$.

(4) $$\mathrm{Ric}(V,W) = \mathrm{Ric}_{F_i}(V,W) - \left(\frac{\Delta_B b_i}{b_i} + (s_i - 1)\frac{\|\mathrm{grad}_B b_i\|_B^2}{b_i^2} + \sum_{k=1,k\neq i}^{m} s_k \frac{g_B(\mathrm{grad}_B b_i, \mathrm{grad}_B b_k)}{b_i b_k} \right) g(V,W) \text{ if } i = j.$$

Now, we will compute the *scalar curvature* of a *multiply warped product*. In order to do that, one can use the following *orthonormal frame* on M constructed as follows.

Let $\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^r} \right\}$ and $\left\{ \frac{\partial}{\partial y_i^1}, \ldots, \frac{\partial}{\partial y_i^{s_i}} \right\}$ be orthonormal frames on open sets $U \subseteq B$ and $V_i \subseteq F_i$, respectively, for any $i \in \{1, \ldots, m\}$. Then

$$\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^r}, \frac{\partial}{b_1 \partial y_1^1}, \ldots, \frac{\partial}{b_1 \partial y_1^{s_1}}, \ldots, \frac{\partial}{b_m \partial y_m^1}, \ldots, \frac{\partial}{b_m \partial y_m^{s_m}} \right\}$$

is an orthonormal frame on an open set $W \subseteq B \times F$ contained in $U \times V \subseteq B \times F$, where $F = F_1 \times \cdots \times F_m$.

Proposition 2.6. *Let $M = B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a pseudo-Riemannian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$. Then, τ admits the following expressions*:

(1)

$$\tau = \tau_B - 2\sum_{i=1}^{m} s_i \frac{\Delta_B b_i}{b_i} + \sum_{i=1}^{m} \frac{\tau_{F_i}}{b_i^2} - \sum_{i=1}^{m} s_i(s_i - 1)\frac{\|\mathrm{grad}_B b_i\|_B^2}{b_i^2} - \sum_{i=1}^{m} \sum_{k=1,k\neq i}^{m} s_k s_i \frac{g_B(\mathrm{grad}_B b_i, \mathrm{grad}_B b_k)}{b_i b_k},$$

(2)

$$\tau = \tau_B - \sum_{i=1}^{m} s_i \frac{\Delta_B b_i}{b_i} - \mathrm{div} \sum_{i=1}^{m} s_i \frac{\mathrm{grad}_B b_i}{b_i} - g_B\left(\sum_{i=1}^{m} s_i \frac{\mathrm{grad}_B b_i}{b_i}, \sum_{i=1}^{m} s_i \frac{\mathrm{grad}_B b_i}{b_i} \right) + \sum_{i=1}^{m} \frac{\tau_{F_i}}{b_i^2}.$$

The following formula can be directly obtained from the previous result and noting that on a multiply generalized Robertson–Walker space–time $\mathrm{grad}_B b_i = -b_i'$, $\|\mathrm{grad}_B b_i\|_B^2 = -(b_i')^2$, $g_B\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = -1$, $\mathrm{H}_B^{b_i}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = b_i''$ and $\Delta_B b_i = -b_i''$, we denote the usual derivative on the real interval I by the prime notation (i.e., $'$) from now on.

Corollary 2.7. *Let $M = I \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a multiply generalized Robertson–Walker space–time with the metric $g = -\mathrm{d}t^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$. Then, τ admits the following expressions*:

(1) $$\tau = 2\sum_{i=1}^{m} s_i \frac{b_i''}{b_i} + \sum_{i=1}^{m} \frac{\tau_{F_i}}{b_i^2} + \sum_{i=1}^{m} s_i(s_i-1)\frac{(b_i')^2}{b_i^2} + \sum_{i=1}^{m}\sum_{k=1,k\neq i}^{m} s_k s_i \frac{b_i' b_k'}{b_i b_k},$$

(2) $$\tau = \sum_{i=1}^{m} s_i \frac{b_i''}{b_i} + \left(\sum_{i=1}^{m} s_i \frac{b_i'}{b_i}\right)' + \left(\sum_{i=1}^{m} s_i \frac{b_i'}{b_i}\right)^2 + \sum_{i=1}^{m} \frac{\tau_{F_i}}{b_i^2}.$$

We now give some physical examples of relativistic space–times and state some of their geometric properties to stress the physical motivation and importance of Lorentzian multiply warped products. The first example is Schwarzschild black hole solution or known as inner Reissner–Nordström space–time and the second one is Kasner space–time. Our last two examples are closely related to each other, more explicitly, the third example is Bañados–Teitelboim–Zanelli (BTZ) black hole solution and the final example is de Sitter (dS) black hole solution.

- **Schwarzschild space–time**

  We will briefly discuss the interior Schwarzschild solution. We show how the interior solution can be written as a multiply warped product.

  The line element of the *Schwarzschild black hole* space–time model for the region $r < 2m$ is given as (see [41])

$$\mathrm{d}s^2 = -\left(\frac{2m}{r} - 1\right)^{-1} \mathrm{d}r^2 + \left(\frac{2m}{r} - 1\right) \mathrm{d}t^2 + r^2\,\mathrm{d}\Omega^2,$$

  where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$ on $\mathbb{S}^2$.

  In [19], it is shown that this space–time model can be expressed as a multiply generalized Robertson–Walker space–time, i.e.,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + b_1^2(t)\,\mathrm{d}r^2 + b_2^2(t)\,\mathrm{d}\Omega^2,$$

  where $b_1(t) = \sqrt{\frac{2m}{F^{-1}(t)} - 1}$ and $b_2(t) = F^{-1}(t)$, also $t = F(r) = 2m\arccos\left(\sqrt{\frac{2m-r}{2m}}\right) - \sqrt{r(2m-r)}$ such that $\lim_{r\to 2m} F(r) = m\pi$ and $\lim_{r\to 0} F(r) = 0$.

  Moreover, we also need to impose the above multiply generalized Robertson–Walker space–time model for the *Schwarzschild black hole* to be Ricci-flat due to the fact that the

Schwarzschild black hole is Ricci-flat (see also the review of Miguel Sánchez in AMS for [19]).

- **Kasner space–time**

  We consider the *Kasner* space–time as a *Lorentzian* multiply warped product (see [61]).

  A Lorentzian multiply warped product (M, g) of the form $M = (0, \infty) \times_{t^{p_1}} \mathbb{R} \times_{t^{p_2}} \mathbb{R} \times_{t^{p_3}} \mathbb{R}$ with the metric $g = -\mathrm{d}t^2 \oplus t^{2p_1}\,\mathrm{d}x^2 \oplus t^{2p_2}\,\mathrm{d}y^2 \oplus t^{2p_3}\,\mathrm{d}z^2$ is said to be the *Kasner* space–time if $p_1 + p_2 + p_3 = (p_1)^2 + (p_2)^2 + (p_3)^2 = 1$ (see [46]).

  It is known by [40] that $-\frac{1}{3} \leq p_1, p_2, p_3 < 1$. It is also known that, excluding the case of two p_i 's zero, then one p_i is negative and the other two are positive. Thus we may assume that $-\frac{1}{3} \leq p_1 < 0 < p_2 \leq p_3 < 1$ by excluding the case of two p_i 's zero and one p_i equal to 1. Furthermore, the only solution in which $p_2 = p_3$ is given by $p_1 = -\frac{1}{3}$ and $p_2 = p_3 = \frac{2}{3}$. Note also that since $-\frac{1}{3} \leq p_1, p_2, p_3 < 1$, we have to assume B to be $(0, \infty)$. Clearly, the *Kasner* space–time is *globally hyperbolic* (see [72]).

  By making use of the results in [72], it can be easily seen that the *Kasner* space–time is future-directed time-like and future-directed null geodesic complete but it is past-directed time-like and past-directed null geodesic incomplete. Moreover, it is also space-like geodesic incomplete.

  Notice that the Kasner space–time is Einstein with $\lambda = 0$ (i.e., Ricci-flat) (see [46] and p. 135 of [56]) and hence has constant scalar curvature as zero. This fact can be proved as a particular consequence of our results in the next section, namely by using Theorem 3.3.

- **Static Bañados–Teitelboim–Zanelli (BTZ) space–time**

  In [42], authors classify (BTZ) black hole solutions into three different classes as static, rotating and charged. Here, we will only give a brief description of a static BTZ space–time in terms of Lorentzian multiply warped products, i.e., multiply generalized Robertson–Walker space–times (see also [8,9,59]). The line element of a static BTZ black hole solution can be expressed as

$$\mathrm{d}s^2 = -N^{-2}\,\mathrm{d}r^2 + N^2\,\mathrm{d}t^2 + r^2\,\mathrm{d}\Omega^2,$$

  where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$ on $\mathbb{S}^2$.

  The line element of the *Static BTZ black hole* space–time model for the region $r < r_H$ can be obtained by taking

$$N^2 = m - \frac{r^2}{l^2}.$$

  In this case, the space–time model can be expressed as a multiply generalized Robertson–Walker space–time, i.e.,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + b_1^2(t)\,\mathrm{d}r^2 + b_2^2(t)\,\mathrm{d}\Omega^2,$$

  where $r_H = l\sqrt{m}$, $b_1(t) = \sqrt{m - \frac{(F^{-1}(t))^2}{l^2}}$ and $b_2(t) = F^{-1}(t)$, also $t = F(r) = l\arcsin\left(\frac{r}{r_H}\right)$ such that $\lim_{r \to r_H} F(r) = \frac{l\pi}{2}$ and $\lim_{r \to 0} F(r) = 0$.

Here, note that the constant scalar curvature τ of the multiply generalized Robertson–Walker space–time introduced above is $\tau = -6/l^2$ (see [42]) or apply Corollary 2.7.

Note that, in [42], they also classify (dS) black hole solution into three classes as static, rotating and charged, similar to (BTZ) black hole solutions (see [8,9,59]).

We now state a couple of results which we will frequently be applied along this article.

The first one is an easy computation which we will show explicitly below. Let (M, g) be an n-dimensional pseudo-Riemannian manifold. For any $t \in \mathbb{R}$ and $v \in C^{\infty}_{>0}(B) = \{v \in C^{\infty}(B) : v > 0\}$:

$$\operatorname{grad}_g v^t = t v^{t-1} \operatorname{grad}_g v, \qquad \Delta_g v^t = t[(t-1)v^{t-2}\|\operatorname{grad}_g v\|_g^2 + v^{t-1}\Delta_g v],$$

$$\frac{\Delta_g v^t}{v^t} = t\left[(t-1)\frac{\|\operatorname{grad}_g v\|_g^2}{v^2} + \frac{\Delta_g v}{v}\right]. \tag{2.2}$$

The second one is a lemma that follows (for a proof and some extensions as well as other useful applications, see Section 2 of [26]).

Lemma 2.8. *Let (M, g) be an n-dimensional pseudo-Riemannian manifold. Let L_g be a differential operator on $C^{\infty}_{>0}(M)$ defined by*

$$L_g v = \sum_{i=1}^{k} r_i \frac{\Delta_g v^{a_i}}{v^{a_i}}, \tag{2.3}$$

where $r_i, a_i \in \mathbb{R}$ and $\zeta := \sum_{i=1}^{k} r_i a_i$, $\eta := \sum_{i=1}^{k} r_i a_i^2$. Then,

(i) $$L_g v = (\eta - \zeta)\frac{\|\operatorname{grad}_g v\|_g^2}{v^2} + \zeta\frac{\Delta_g v}{v}. \tag{2.4}$$

(ii) *If $\zeta \neq 0$ and $\eta \neq 0$, for $\alpha = \frac{\zeta}{\eta}$ and $\beta = \frac{\zeta^2}{\eta}$, then we have*

$$L_g v = \beta\frac{\Delta_g v^{1/\alpha}}{v^{1/\alpha}}. \tag{2.5}$$

3. Special multiply warped products

3.1. Einstein–Ricci tensor

In this section, we state some condition to guarantee that a *multiply generalized Robertson–Walker space–time* is *Ricci-flat* or *Einstein*.

Now, we recall some elementary facts about Einstein manifolds starting from its definition.

Recall that an n-dimensional pseudo-Riemannian manifold (M, g) is said to be Einstein if there exists a smooth real-valued function λ on M such that $\mathrm{Ric} = \lambda g$, and λ is called the Ricci curvature of (M, g) (see also p. 7 of [6]).

Remark 3.1. Concerning to this notion, it should be pointed out:

(1) If (M, g) is Einstein and $n \geq 3$, then λ is constant and $\lambda = \tau/n$, where τ is the constant scalar curvature of (M, g).
(2) If (M, g) is Einstein and $n = 2$, then λ is not necessarily constant.
(3) If (M, g) has constant sectional curvature k, then (M, g) is Einstein with $\lambda = k(n-1)$ and has constant scalar curvature $\tau = n(n-1)k$.
(4) (M, g) is Einstein with Ricci curvature λ and $n = 3$, then (M, g) is a space of constant (sectional) curvature $\mathrm{K} = \lambda/2$.
(5) If (M, g) is a Lorentzian manifold then (M, g) is Einstein if and only if $\mathrm{Ric}(v, v) = 0$, for any null vector field v on M.

By using Proposition 2.5, we easily obtain the *Ricci curvature* of *Lorentzian multiply warped products*, (M, g) of the above form.

Proposition 3.2. *Let $M = I \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a multiply generalized Robertson–Walker space–time with the metric $g = -\mathrm{d}t^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$ also let $\frac{\partial}{\partial t} \in \mathfrak{X}(I)$ and $v_i \in \mathfrak{X}(F_i)$, for any $i \in \{1, \ldots, m\}$. If $v = \sum_{i=1}^m v_i \in \mathfrak{X}(F)$, then*

$$\mathrm{Ric}\left(\frac{\partial}{\partial t} + v, \frac{\partial}{\partial t} + v\right) = \sum_{i=1}^{m} \left(\mathrm{Ric}_{F_i}(v_i, v_i) + \left(b_i b_i'' + (s_i - 1)(b_i')^2 + b_i b_i' \sum_{k=1, k \neq i}^{m} s_k \frac{b_k'}{b_k} \right) \times g_{F_i}(v_i, v_i) - s_i \frac{b_i''}{b_i} \right).$$

Proof. By substituting $\overline{X} = \frac{\partial}{\partial t} + \sum_{i=1}^m v_i$ and $\overline{Y} = \frac{\partial}{\partial t} + \sum_{i=1}^m v_i$ and by noting that $\mathrm{grad}_B b_i = -b_i'$, $g_B\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = -1$, $g_B(\mathrm{grad}_B b_i, \mathrm{grad}_B b_i) = -(b_i')^2$, $\mathrm{H}_B^{b_i}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = b_i''$, $\Delta_B b_i = -b_i''$ and $\mathrm{Ric}_B\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = 0$ and by using Proposition 2.5, we obtain the result.

The following result can be easily proved by substituting $v_j = 0$ for any $j \in \{1, \ldots, m\} \backslash \{i\}$ and $v_i \neq 0$, in Proposition 3.2 along with the method of separation of variables.

Theorem 3.3. *Let $M = I \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a multiply generalized Robertson–Walker space–time with the metric $g = -\mathrm{d}t^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$. The space–time (M, g) is Einstein with Ricci curvature λ if and only if the following conditions are satisfied for any $i \in \{1, \ldots, m\}$*

(1) *each fiber (F_i, g_{F_i}) is Einstein with Ricci curvature λ_{F_i} for any $i \in \{1, \ldots, m\}$,*
(2) $\sum_{i=1}^m s_i \frac{b_i''}{b_i} = \lambda$ *, and*
(3) $\lambda_{F_i} + b_i b_i'' + (s_i - 1)(b_i')^2 + b_i b_i' \sum_{k=1, k \neq i}^m s_k \frac{b_k'}{b_k} = \lambda b_i^2$.

Remark 3.4. In Theorem 3.3, Eq. (3) can be expressed in different forms and here we want to present some of them. By applying Eq. (2.2), we can have

$$\frac{\lambda_{F_i}}{b_i^2} + \frac{1}{s_i}\frac{(b_i^{s_i})''}{b_i^{s_i}} + \frac{b_i'}{b_i}\sum_{k=1,k\neq i}^{m} s_k \frac{b_k'}{b_k} = \lambda, \qquad (E_{gRW}\text{-i})$$

or equivalently:

$$\frac{\lambda_{F_i}}{b_i^2} + \frac{b_i''}{b_i} - \frac{(b_i')^2}{b_i^2} + \frac{b_i'}{b_i}\sum_{k=1}^{m} s_k \frac{b_k'}{b_k} = \lambda. \qquad (E_{gRW}\text{-ii})$$

3.2. Constant scalar curvature

It is possible to obtain equivalent expressions for the scalar curvature in Corollary 2.7, namely the following just follows from Eq. (2.2):

$$\tau = \sum_{i=1}^{m}\left[s_i \frac{b_i''}{b_i} + \frac{(b_i^{s_i})''}{b_i^{s_i}}\right] + \sum_{i=1}^{m}\frac{\tau_{F_i}}{b_i^2} + \sum_{i=1}^{m}\sum_{k=1,k\neq i}^{m} s_k s_i \frac{b_i'}{b_i}\frac{b_k'}{b_k}. \qquad (sc_{gRW}\text{-i})$$

Since $2s_i \neq 0$ and $s_i + s_i^2 = s_i(s_i+1) \neq 0$, by Lemma 2.8, there results

$$\tau = \sum_{i=1}^{m}\frac{4s_i}{s_i+1}\frac{(b_i^{(s_i+1)/2})''}{b_i^{(s_i+1)/2}} + \sum_{i=1}^{m}\frac{\tau_{F_i}}{b_i^2} + \sum_{i=1}^{m}\sum_{k=1,k\neq i}^{m} s_k s_i \frac{b_i'}{b_i}\frac{b_k'}{b_k}. \qquad (sc_{gRW}\text{-ii})$$

Thus, defining $\psi_i = b_i^{(s_i+1)/2}$, results

$$\tau = \sum_{i=1}^{m}\frac{4s_i}{s_i+1}\frac{\psi_i''}{\psi_i} + \sum_{i=1}^{m}\frac{\tau_{F_i}}{\psi_i^{4/(s_i+1)}} + \sum_{i=1}^{m}\sum_{k=1,k\neq i}^{m} s_k s_i \frac{(\psi_i^{2/(s_i+1)})'}{\psi_i^{2/(s_i+1)}}\frac{(\psi_k^{2/(s_k+1)})'}{\psi_k^{2/(s_k+1)}}. \qquad (sc_{gRW}\text{-iii})$$

Note that when $m = 1$ this relation is exactly that obtained in [24,26] when the base has dimension 1.

The following result just follows from the method of separation of variables and the fact that each $\tau_{F_i} : F_i \to \mathbb{R}$ is a function defined on F_i , for any $i \in \{1, \ldots, m\}$.

Proposition 3.5. *Let $M = I \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a multiply generalized Robertson–Walker space–time with the metric $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$. If the space–time (M, g) has constant scalar curvature τ , then each fiber (F_i, g_{F_i}) has constant scalar curvature τ_{F_i} , for any $i \in \{1, \ldots, m\}$.*

As one can notice from the above formula, it is extremely hard to determine general solutions for warping functions which produce an Einstein, or with constant scalar curvature multiply generalized Robertson–Walker space–time. Note that non-linear second order differential equations need to be solved according Theorem 3.3. Further note that there is only one differential equation and m different warping functions in Corollary 2.7. Therefore instead of giving a general answer to the existence of warping functions to get an Einstein, or with constant scalar curvature, space–time, we simplify this problem and consider some specific cases in mentioned Sections 4 and 5.

4. Generalized Kasner space–time

In this section we give an extension of Kasner space–times and consider their scalar and Ricci curvatures.

Definition 4.1. A generalized Kasner space–time (M, g) is a Lorentzian multiply warped product of the form $M = I \times_{\varphi^{p_1}} F_1 \times \cdots \times_{\varphi^{p_m}} F_m$ with the metric $g = -\mathrm{d}t^2 \oplus \varphi^{2p_1} g_{F_1} \oplus \cdots \oplus \varphi^{2p_m} g_{F_m}$, where $\varphi : I \to (0, \infty)$ is smooth and $p_i \in \mathbb{R}$, for any $i \in \{1, \ldots, m\}$ and also $I = (t_1, t_2)$ with $-\infty \le t_1 < t_2 \le \infty$.

Notice that a Kasner space–time can be obtained out of a form defined above by taking $\varphi = Id_{(0,\infty)}$ with $m = 3$ and $I = (0, \infty)$, where $Id_{(0,\infty)}$ denotes for the identity function on $(0, \infty)$ (see [40]).

From now on, for an arbitrary generalized Kasner space–time of the form in Definition 4.1, we introduce the following parameters

$$\zeta := \sum_{l=1}^{m} s_l p_l \quad \text{and} \quad \eta := \sum_{l=1}^{m} s_l p_l^2. \tag{$\zeta;\eta$}$$

Remark 4.2. Note that $\zeta \neq 0$ implies $\eta \neq 0$ and in this case, defining $S = \sum_{l=1}^{m} s_l$, results $\frac{\eta}{\zeta^2} \ge \frac{1}{S}$. The latter is for example consequence of the Hölder inequality (compare with p. 186 of [35]).

By applying Theorem 3.3, we can easily state the following result and later we will examine the solvability of the differential equations therein.

Proposition 4.3. *Let $M = I \times_{\varphi_1} F_1 \times \cdots \times_{\varphi_m} F_m$ be a generalized Kasner space–time with the metric $g = -\mathrm{d}t^2 \oplus \varphi^{2p_1} g_{F_1} \oplus \cdots \oplus \varphi^{2p_m} g_{F_m}$. Then the space–time (M, g) is Einstein with Ricci curvature λ if and only if*

(1) *each fiber (F_i, g_{F_i}) is Einstein with Ricci curvature λ_{F_i} for any $i \in \{1, \ldots, m\}$,*
(2) $\lambda = \sum_{l=1}^{m} s_l \frac{(\varphi^{p_l})''}{\varphi^{p_l}} = (\eta - \zeta)\frac{(\varphi')^2}{\varphi^2} + \zeta\frac{\varphi''}{\varphi}$ *, and*
(3) $\frac{\lambda_{F_i}}{\varphi^{2p_i}} + p_i\left[(\zeta - 1)\frac{(\varphi')^2}{\varphi^2} + \frac{\varphi''}{\varphi}\right] = \lambda$.

Remark 4.4. Moreover, if in Proposition 4.3 we assume that $\zeta \neq 0$ also, then by Remark 4.2 is $\eta \neq 0$. Hence, (3) is equivalent to

$$\frac{\lambda_{F_i}}{\varphi^{2p_i}} + \frac{p_i}{\zeta}\frac{(\varphi^{\zeta})''}{\varphi^{\zeta}} = \lambda, \qquad (E_{gK}^{(3)}\text{-i})$$

and (2) is equivalent to

$$\lambda = \frac{\zeta^2}{\eta}\frac{(\varphi^{\frac{\eta}{\zeta}})''}{\varphi^{\frac{\eta}{\zeta}}}. \qquad (E_{gK}^{(2)}\text{-i})$$

Proof (of Proposition 4.3 and Remark 4.4). In order to prove (3), note that Eq. (E_{gRW}-i) says

$$\frac{\lambda_{F_i}}{\varphi^{2p_i}} + \frac{1}{s_i}\frac{(\varphi^{p_i s_i})''}{\varphi^{p_i s_i}} + \frac{(\varphi^{p_i})'}{\varphi^{p_i}}\sum_{k=1,k\neq i}^{m} s_k\frac{(\varphi^{p_k})'}{\varphi^{p_k}} = \lambda.$$

Hence, by Eq. (2.2):

$$\frac{\lambda_{F_i}}{\varphi^{2p_i}} + p_i(p_i s_i - 1)\frac{(\varphi')^2}{\varphi^2} + p_i\frac{\varphi''}{\varphi} + p_i\frac{\varphi'}{\varphi}\sum_{k=1,k\neq i}^{m} s_k p_k\frac{\varphi'}{\varphi} = \lambda$$

and from here

$$\frac{\lambda_{F_i}}{\varphi^{2p_i}} + p_i\left[(p_i s_i - 1) + \sum_{k=1,k\neq i}^{m} s_k p_k\right]\frac{(\varphi')^2}{\varphi^2} + p_i\frac{\varphi''}{\varphi} = \lambda.$$

So

$$\frac{\lambda_{F_i}}{\varphi^{2p_i}} + p_i\left[\left(-1 + \sum_{k=1}^{m} s_k p_k\right)\frac{(\varphi')^2}{\varphi^2} + \frac{\varphi''}{\varphi}\right] = \lambda$$

and by the definition of ζ

$$\frac{\lambda_{F_i}}{\varphi^{2p_i}} + p_i\left[(\zeta - 1)\frac{(\varphi')^2}{\varphi^2} + \frac{\varphi''}{\varphi}\right] = \lambda.$$

If furthermore $\zeta \neq 0$, applying again Eq. (2.2), results ($E_{gK}^{(3)}$-i).

On the other hand, from (2) of Theorem 3.3

$$\lambda = \sum_{l=1}^{m} s_l\frac{(\varphi^{p_l})''}{\varphi^{p_l}}$$

and by Lemma 2.8(a):

$$\lambda = (\eta - \zeta)\frac{(\varphi')^2}{\varphi^2} + \zeta\frac{\varphi''}{\varphi}.$$

Hence, if $\zeta \neq 0$ and as consequence $\eta \neq 0$, applying Lemma 2.8(b), results ($E^{(2)}_{gK}$-i).

Note that, from now on and also including the previous result, when we apply Lemma 2.8, we denote the usual derivative in equations by means of the prime notation.

Remark 4.5. Note that the conditions $\zeta \neq 0$ and $\eta \neq 0$ agree with the conditions usually imposed in the classical Kasner space–times, namely $p_1 + p_2 + p_3 = 1$ and $p_1^2 + p_2^2 + p_3^2 = 1$ (see [46]). It is easy to show that the unique possibility to construct an Einstein classical Kasner manifold or a constant scalar curvature classical Kasner manifold with $p_1 + p_2 + p_3 = 0$ is $p_1 = p_2 = p_3 = 0$, so that we have just a usual product. Indeed, considering $\varphi(t) = t$, it is possible to apply Proposition 4.3 and later Proposition 4.11, respectively.

Corollary 4.6. *Under the hypothesis of Proposition 4.3, along with $\zeta \neq 0$ and $\eta \neq 0$. Assume also that for all i, $\zeta - p_i \neq 0$ and $\eta - p_i\zeta \neq 0$. Then, M is Einstein if and only if for any $i \in \{1, \dots, m\}$, (F_i, g_{F_i}) is Einstein Ricci curvature λ_{F_i} and*

$$\frac{(\zeta - p_i)^2}{\eta - p_i\zeta}\frac{\psi''}{\psi} = \frac{\lambda_{F_i}}{\psi^{((\zeta-p_i)/(\eta-p_i\zeta))2p_i}}, \tag{4.1}$$

where $0 < \psi := \varphi^{(\eta-p_i\zeta)/(\eta-p_i)}$.

Proof. Indeed, from equations ($E^{(3)}_{gK}$-i) and ($E^{(2)}_{gK}$-i):

$$\frac{\lambda_{F_i}}{\varphi^{2p_i}} = \frac{\zeta^2}{\eta}\frac{(\varphi^{\eta/\zeta})''}{\varphi^{\eta/\zeta}} - \frac{p_i}{\zeta}\frac{(\varphi^{\zeta})''}{\varphi^{\zeta}}.$$

Thus, since for all i, $\zeta - p_i \neq 0$ and $\eta - p_i\zeta \neq 0$, then applying Lemma 2.8, the result just follows.

Example 4.7. Under the conditions of the classical Kasner metrics, $m = 3$, $p_1 + p_2 + p_3 = 1$ and $p_1^2 + p_2^2 + p_3^2 = 1$, we have $\lambda_{F_i} = 0$, $\zeta = 1$ and $\eta = 1$. Hence the hypothesis $\zeta - p_i \neq 0$ and $\eta - p_i\zeta \neq 0$, for all i, implies that $p_i \neq 1$ for all i. In this case, Eq. (4.1) is equivalent to $0 < \psi = \varphi$ and $\psi'' = 0$, i.e., $0 < \varphi(t) = at + b$ with $a, b \geq 0$ and $a^2 + b^2 \neq 0$. Hence, from Eq. ($E^{(2)}_{gK}$-i), $(0, +\infty) \times_{\varphi^{p_1}} \mathbb{R} \times_{\varphi^{p_2}} \mathbb{R} \times_{\varphi^{p_3}} \mathbb{R}$ is Ricci flat space–time.

Corollary 4.8. *Let us assume the hypothesis of Corollary 4.6 and that for all i, (F_i, g_{F_i}) is Ricci flat. Then, M is Einstein if and only if $\psi'' = 0$ with*

$$0 < \psi := \varphi^{(\eta-p_i\zeta)/(\eta-p_i)} \quad \textit{for all } i.$$

Proof. It is an immediate consequence of Corollary 4.6.

Corollary 4.9. *Assume that* (F_i, g_{F_i}) *is Ricci flat for all i. Let also* $\overline{\zeta}, \overline{\eta} \in \mathbb{R} \setminus \{0\}$ *such that* $\overline{\zeta}^2 = \overline{\eta}$ *and* $\psi(t) = at + b$ *with* $a, b \geq 0$ *and* $a^2 + b^2 > 0$. *If* $\zeta = \overline{\zeta}$, $\eta = \overline{\eta}$, $\zeta - p_i \neq 0$ *and* $\eta - p_i\zeta \neq 0$ *for all i, then* $M = (0, \infty) \times_{\varphi^{p_1}} F_1 \times \cdots \times_{\varphi^{p_m}} F_m$ *is a Ricci flat space–time, where* $\varphi = \psi^{1/\zeta}$.

Proof. It is sufficient to apply Corollary 4.8 and Proposition 4.3.

Remark 4.10. Note that Corollary 4.9 contains the classical Kasner metrics except the case in which at least one $p_i = 1$ (really at most one could be 1 because $\eta = p_1^2 + p_2^2 + p_3^2 = 1$).

The following just follows from Corollary 2.7 and again we discuss the existence of a solution for the differential equation below.

Proposition 4.11. *Let* $M = I \times_{\varphi_1} F_1 \times \cdots \times_{\varphi_m} F_m$ *be a generalized Kasner space–time with the metric* $g = -\mathrm{d}t^2 \oplus \varphi^{2p_1} g_{F_1} \oplus \cdots \oplus \varphi^{2p_m} g_{F_m}$. *Then the space–time* (M, g) *has constant scalar curvature* τ *if and only if*

(1) *each fiber* (F_i, g_{F_i}) *has constant scalar curvature* τ_{F_i} *for any* $i \in \{1, \ldots, m\}$, *and*
(2) $\tau = 2\zeta\frac{\varphi''}{\varphi} + [(\zeta - 2)\zeta + \eta]\frac{(\varphi')^2}{\varphi^2} + \sum_{i=1}^m \frac{\tau_{F_i}}{\varphi^{2p_i}}$,

Remark 4.12. If $\zeta \neq 0$, then (2) in Proposition 4.11 is equivalent to

$$\tau = \frac{4\zeta^2}{\zeta^2 + \eta} \frac{(\varphi^{(\zeta^2+\eta)/(2\zeta)})''}{\varphi^{(\zeta^2+\eta)/(2\zeta)}} + \sum_{i=1}^m \frac{\tau_{F_i}}{\varphi^{2p_i}}.$$

Proof of Proposition 4.11 and Remark 4.12. For each $i \in \{1, \ldots, m\}$, let $\gamma_i = p_i\frac{s_i+1}{2}$ and $\psi_i = \varphi^{\gamma_i}$, then by (sc_{gRW}-iii) and Eq. (2.2) there results

$$\tau = \sum_{i=1}^m \frac{4s_i}{s_i + 1}\gamma_i \left[(\gamma_i - 1)\frac{(\varphi')^2}{\varphi^2} + \frac{\varphi''}{\varphi}\right] + \sum_{i=1}^m \frac{\tau_{F_i}}{\varphi^{(4/(s_i+1))\gamma_i}}$$
$$+ \sum_{i=1}^m \sum_{k=1,k\neq i}^m s_k s_i \frac{2\gamma_i}{s_i + 1} \frac{2\gamma_k}{s_k + 1} \frac{(\varphi')^2}{\varphi^2}.$$

Then we have

$$\tau = \sum_{i=1}^m 2s_i p_i \left[\left(p_i\frac{s_i + 1}{2} - 1\right) \frac{(\varphi')^2}{\varphi^2} + \frac{\varphi''}{\varphi}\right] + \sum_{i=1}^m \frac{\tau_{F_i}}{\varphi^{2p_i}}$$
$$+ \sum_{i=1}^m \sum_{k=1,k\neq i}^m s_k s_i p_i p_k \frac{(\varphi')^2}{\varphi^2}$$

$$= 2\zeta\frac{\varphi''}{\varphi} + \sum_{i=1}^{m} s_i p_i \left[2\left(p_i\frac{s_i+1}{2} - 1\right) + \sum_{k=1,k\neq i}^{m} s_k p_k\right]\frac{(\varphi')^2}{\varphi^2} + \sum_{i=1}^{m}\frac{\tau_{F_i}}{\varphi^{2p_i}}$$

$$= 2\zeta\frac{\varphi''}{\varphi} + \sum_{i=1}^{m} s_i p_i[(\zeta-2)+p_i]\frac{(\varphi')^2}{\varphi^2} + \sum_{i=1}^{m}\frac{\tau_{F_i}}{\varphi^{2p_i}}$$

$$= 2\zeta\frac{\varphi''}{\varphi} + [(\zeta-2)\zeta+\eta]\frac{(\varphi')^2}{\varphi^2} + \sum_{i=1}^{m}\frac{\tau_{F_i}}{\varphi^{2p_i}}.$$

Since $(\zeta-2)\zeta+\eta+1 = (\zeta-1)^2+\eta = 0$ if and only if $p_i = 0$ for all $i \in \{1,\dots,m\}$, if at least one $p_i \neq 0$ there results by Eq. (2.2)

$$\tau = (2\zeta-1)\frac{\varphi''}{\varphi} + \frac{1}{(\zeta-1)^2+\eta}\frac{(\varphi^{(\zeta-1)^2+\eta})''}{\varphi^{(\zeta-1)^2+\eta}} + \sum_{i=1}^{m}\frac{\tau_{F_i}}{\varphi^{2p_i}}.$$

Hence, if $\zeta \neq 0$, applying Lemma 2.8:

$$\tau = \frac{4\zeta^2}{\zeta^2+\eta}\frac{(\varphi^{(\zeta^2+\eta)/(2\zeta)})''}{\varphi^{(\zeta^2+\eta)/(2\zeta)}} + \sum_{i=1}^{m}\frac{\tau_{F_i}}{\varphi^{2p_i}}.$$

Corollary 4.13. *Under the hypothesis of* Proposition 4.11 *and* $\zeta \neq 0$. *Then, by changing variables as* $u = \varphi^{(\zeta^2+\eta)/(2\zeta)}$, *we conclude that the space–time M has constant scalar curvature* τ *if and only if*

$$\tau = \frac{4\zeta^2}{\zeta^2+\eta}\frac{u''}{u} + \sum_{i=1}^{m}\frac{\tau_{F_i}}{u^{4\zeta/(\zeta^2+\eta)p_i}}$$

or equivalently

$$-\frac{4}{1+\frac{\eta}{\zeta^2}}u'' = -\tau u + \sum_{i=1}^{m}\tau_{F_i}u^{1-4/(1+\eta/\zeta^2)(p_i/\zeta)}.$$

Remark 4.14. If $\zeta \neq 0$ and there is only one fiber, i.e., in a standard warped product, the equation in the previous corollary corresponds to those obtained in [24,26].

Example 4.15. Let us assume that $\zeta \neq 0$ and each F_i is scalar flat, namely $\tau_{F_i} = 0$. Hence, equation in the previous corollary is written as

$$-\frac{4\zeta^2}{\zeta^2+\eta}u'' = -\tau u.$$

Thus all the solutions have the form

$$u(t)=\begin{cases}\mathcal{A}\,\mathrm{e}^{\mathrm{i}\sqrt{-\tau\frac{\zeta^2+\eta}{2\zeta^2}}t}+\mathcal{B}\,\mathrm{e}^{-\mathrm{i}\sqrt{-\tau\frac{\zeta^2+\eta}{2\zeta^2}}t} & \text{if } \tau<0,\\ \mathcal{A}t+\mathcal{B} & \text{if } \tau=0,\\ \mathcal{A}\,\mathrm{e}^{\sqrt{-\tau\frac{\zeta^2+\eta}{2\zeta^2}}t}+\mathcal{B}\,\mathrm{e}^{\sqrt{-\tau\frac{\zeta^2+\eta}{2\zeta^2}}t} & \text{if } \tau>0\end{cases}$$

with constants $\mathcal{A}$ and $\mathcal{B}$ such that $u>0$.

If $\zeta=0$, by Proposition 4.11, we look for positive solutions of the equation

$$\tau=\eta\frac{(\varphi')^2}{\varphi^2},\quad \varphi>0.$$

Since $\eta>0$, the latter is equivalent to

$$\left(\varphi\frac{\sqrt{\tau}}{\sqrt{\eta}}+\varphi'\right)\left(\varphi\frac{\sqrt{\tau}}{\sqrt{\eta}}-\varphi'\right)=0,\quad \varphi>0.$$

Solutions of the equation above are given as:

$$\varphi(t)=C\,\mathrm{e}^{\pm(\sqrt{\tau}/\sqrt{\eta})t},$$

where C is a positive constant.

Note that this example include the situation of the classical Kasner space–times in the framework of scalar curvature. Compare with the results about Einstein classical Kasner metrics in Remark 4.5 and Example 4.7.

5. Four-dimensional space–time models

We first give a classification of four-dimensional warped product space–time models and then consider Ricci tensors and scalar curvatures of them.

Definition 5.1. Let $M=I\times_{b_1}F_1\times\cdots\times_{b_m}F_m$ be a multiply generalized Robertson–Walker space–time with metric $g=-\mathrm{d}t^2\oplus b_1^2g_{F_1}\oplus\cdots\oplus b_m^2g_{F_m}$.

- (M,g) is said to be of Type (I) if $m=1$ and $\dim(F)=3$.
- (M,g) is said to be of Type (II) if $m=2$ and $\dim(F_1)=1$ and $\dim(F_2)=2$.
- (M,g) is said to be of Type (III), if $m=3$ and $\dim(F_1)=1$, $\dim(F_2)=1$ and $\dim(F_3)=1$.

Note that Type (I) contains the Robertson–Walker space–time. The Schwarzschild black hole solution can be considered as an example of Type (II). Type (III) includes the Kasner space–time.

5.1. Type (I)

Let $M = I \times_b F$ be a Type (I) warped product space–time with metric $g = -\mathrm{d}t^2 \oplus b^2 g_F$. Then the scalar curvature τ of (M, g) is given as

$$\tau = \frac{\tau_F}{b^2} + 6\left(\frac{b''}{b} + \frac{(b')^2}{b^2}\right).$$

The problem of constant scalar curvatures of this type of warped products, known as generalized Robertson–Walker space–times is studied in [29], indeed, explicit solutions to warping function are obtained to have a constant scalar curvature.

If v is a vector field on F and $\overline{x} = \frac{\partial}{\partial t} + v$, then

$$\mathrm{Ric}(\overline{x}, \overline{x}) = \mathrm{Ric}_F(v, v) + (bb'' + 2(b')^2) g_F(v, v) - 3\frac{b''}{b}.$$

In [2], explicit solutions are also obtained for the warping function to make the space–time as Einstein when the fiber is also Einstein.

5.2. Type (II)

Let $M = I \times_{b_1} F_1 \times_{b_2} F_2$ be a Type (II) warped product space–time with metric $g = -\mathrm{d}t^2 \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2}$. Then the scalar curvature τ of (M, g) is given as

$$\tau = \frac{\tau_{F_2}}{b_2^2} + 2\frac{b_1''}{b_1} + 4\frac{b_2''}{b_2} + 2\left(\frac{b_2'}{b_2}\right)^2 + 4\frac{b_1' b_2'}{b_1 b_2}.$$

Note that $\tau_{F_1} = 0$, since $\dim(F_1) = 1$.

If v_i is a vector field on F_i , for any $i \in \{1, 2\}$ and $\overline{x} = \frac{\partial}{\partial t} + v_1 + v_2$, then

$$\mathrm{Ric}(\overline{x}, \overline{x}) = \mathrm{Ric}_{F_2}(v_2, v_2) - \frac{b_1''}{b_1} - \frac{b_2''}{b_2} + \left(b_1 b_1'' + 2\frac{b_1 b_1' b_2'}{b_2}\right) g_{F_1}(v_1, v_1) + \left(b_2 b_2'' + (b_2')^2 + \frac{b_2 b_2' b_1'}{b_1}\right) g_{F_2}(v_2, v_2)$$

Note that $\mathrm{Ric}_{F_1} \equiv 0$, since $\dim(F_1) = 1$.

- **Classification of Einstein Type (II) generalized Kasner space–times**

  Let $M = I \times_{\varphi^{p_1}} F_1 \times_{\varphi^{p_2}} F_2$ be an Einstein Type (II) generalized Kasner space–time. Then the parameters introduced before Proposition 4.3 are given by $\zeta = p_1 + 2p_2$,

$\eta = p_1^2 + 2p_2^2$. Hence the latter arises

$$\begin{cases} (\eta-\zeta)\frac{(\varphi')^2}{\varphi^2} + \zeta\frac{\varphi''}{\varphi} = \lambda, \\ p_1\left[(\zeta-1)\frac{(\varphi')^2}{\varphi^2} + \frac{\varphi''}{\varphi}\right] = \lambda, \\ \frac{\lambda_{F_2}}{\varphi^{2p_2}} + p_2\left[(\zeta-1)\frac{(\varphi')^2}{\varphi^2} + \frac{\varphi''}{\varphi}\right] = \lambda. \end{cases} \tag{E-K-II}$$

The last equation implies in particular that λ_{F_2} is constant.

Let the system

$$(\varphi^\sigma)'' = \nu\varphi^\sigma, \quad 0 < \varphi, \tag{$\varphi^\sigma;\nu$}$$

where ν and σ are real parameters. All its solutions φ^σ have the form

$$\varphi^\sigma(t) = \begin{cases} \mathcal{A}\,\mathrm{e}^{\mathrm{i}\sqrt{-\nu}t} + \mathcal{B}\,\mathrm{e}^{-\mathrm{i}\sqrt{-\nu}t} & \text{if } \nu < 0, \\ \mathcal{A}t + \mathcal{B} & \text{if } \nu = 0, \\ \mathcal{A}\,\mathrm{e}^{\sqrt{\nu}t} + \mathcal{B}\,\mathrm{e}^{-\sqrt{\nu}t} & \text{if } \nu > 0 \end{cases}$$

with constants $\mathcal{A}$ and $\mathcal{B}$ such that $\varphi > 0$.

Furthermore, let the $(\varphi^\sigma;\nu)$ modified system

$$\begin{cases} (\varphi^\sigma)'' = \nu\varphi^\sigma, \\ (\varphi^\sigma)']^2 = \nu(\varphi^\sigma)^2, \\ \varphi > 0. \end{cases} \tag{$\varphi^\sigma;\nu;*$}$$

Note that ν must be > 0 . It is easy to verify that all its solutions are given by

$$\varphi^\sigma(t) = \mathcal{A}\,\mathrm{e}^{\pm\sqrt{\nu}t},$$

where $\mathcal{A}$ is a positive constant.

Consider now two cases, namely

$\underline{\zeta = 0}$: first of all, note that $p_2 = -\frac{1}{2}p_1$ and $\eta = \frac{3}{2}p_1^2$.

$\underline{\eta = 0}$: thus, $p_i = 0$, for all i and $0 = \lambda = \lambda_{F_2}$. Thus the corresponding metric is

$$-\mathrm{d}t^2 + g_{F_1} + g_{F_2}.$$

$\underline{\eta \neq 0}$: then $p_1 \neq 0$, $p_2 \neq 0$ and

$$\begin{cases} \eta\frac{(\varphi')^2}{\varphi^2} = \lambda, \\ p_1\left[-\frac{(\varphi')^2}{\varphi^2} + \frac{\varphi''}{\varphi}\right] = \lambda, \\ \frac{\lambda_{F_2}}{\varphi^{-p_1}} - \frac{1}{2}p_1\left[-\frac{(\varphi')^2}{\varphi^2} + \frac{\varphi''}{\varphi}\right] = \lambda. \end{cases} \tag{E-K-IIi}$$

If

$\underline{\lambda_{F_2} = 0}$: then $\lambda = 0$ and φ is constant φ_0 . Thus the corresponding metric is

$$-\mathrm{d}t^2 + \varphi_0^{2p_1}g_{F_1} + \varphi_0^{2p_2}g_{F_2}.$$

$\underline{\lambda_{F_2} \neq 0}$: then $\frac{\lambda_{F_2}}{\varphi^{-p_1}} = \frac{3}{2}\lambda$, as consequence φ is constant and considering the system this gives a contradiction.

$\underline{\zeta \neq 0}$: hence $\eta \neq 0$ and by Remark 4.4 the system reduces to

$$\begin{cases} \frac{\zeta^2}{\eta}\frac{(\varphi^{\zeta(\eta/\zeta^2)})''}{\varphi^{\zeta(\eta/\zeta^2)}} = \lambda, \\ \frac{p_1}{\zeta}\frac{(\varphi^{\zeta})''}{\varphi^{\zeta}} = \lambda, \\ \frac{\lambda_{F_2}}{\varphi^{2p_2}} + \frac{p_2}{\zeta}\frac{(\varphi^{\zeta})''}{\varphi^{\zeta}} = \lambda. \end{cases} \tag{E-K-IIii }$$

$\underline{\eta = \zeta^2}$: so $p_1 \neq 0$ and either $p_2 = 0$ or $p_2 = -2p_1$. If

$\underline{\lambda = 0}$: then $\lambda_{F_2} = 0$. Thus, the corresponding metric is

$$-\mathrm{d}t^2 + \varphi^{2p_1}g_{F_1} + \varphi^{2p_2}g_{F_2},$$

where φ satisfies $(\varphi^{\zeta}; 0)$.

$\underline{\lambda \neq 0}$: then $\zeta = p_1$ and $p_2 = 0$. Hence, by the third equation $\lambda_{F_2} = \lambda$. Thus, the corresponding metric is

$$-\mathrm{d}t^2 + \varphi^{2\zeta}g_{F_1} + g_{F_2},$$

where φ satisfies $(\varphi^{\zeta}; \lambda)$.

$\underline{\eta \neq \zeta^2}$: then $p_2 \neq 0$ and $\frac{p_1}{p_2}\frac{\lambda_{F_2}}{\varphi^{2p_2}} = \left(\frac{p_1}{p_2} - 1\right)\lambda$. So, if

$\underline{\lambda = 0}$: then the first equation implies, $\varphi^{\zeta} = (\mathcal{A}t + \mathcal{B})^{\zeta^2/\eta}$ and $(\varphi^{\zeta})'' = \frac{\zeta^2}{\eta}\left(\frac{\zeta^2}{\eta} - 1\right)(\mathcal{A}t + \mathcal{B})^{\zeta^2/\eta - 2}\mathcal{A}^2$.

$\underline{\lambda_{F_2} = 0}$: then applying the third equation results $\mathcal{A} = 0$, so φ^{ζ} is constant and φ is a positive constant φ_0 . Thus the corresponding metric is

$$-\mathrm{d}t^2 + \varphi_0^{2p_1}g_{F_1} + \varphi_0^{2p_2}g_{F_2}.$$

$\underline{\lambda_{F_2} \neq 0}$: then $p_1 = 0$, hence $p_2 = \frac{\zeta}{2}$, $\frac{\eta}{\zeta^2} = \frac{1}{2}$. So, by the third equation $\lambda_{F_2} = -\mathcal{A}^2 < 0$. Thus the corresponding metric is

$$-\mathrm{d}t^2 + g_{F_1} + \varphi^{2p_2}g_{F_2}$$

with φ as above.

$\underline{\lambda \neq 0}$: then $p_1 \neq 0$, hence $\frac{\lambda_{F_2}}{\varphi^{2p_2}} = \left(1 - \frac{p_1}{p_2}\right)\lambda$.

$\underline{\lambda_{F_2} = 0}$: then $p_1 = p_2$ and the system can be reduced to

$$3\frac{(\varphi^{\zeta(1/3)})''}{\varphi^{\zeta(1/3)}} = \frac{1}{3}\frac{(\varphi^{\zeta})''}{\varphi^{\zeta}} = \lambda,$$

which is equivalent to the solvable system $(\varphi^{\zeta}; 3\lambda; *)$. Note that λ must be > 0 .

$\underline{\lambda_{F_2} \neq 0}$: then φ is constant and this gives a contradiction.

Table 1 specifies the only possible Einstein generalized Kasner space–times of Type (II) with the corresponding parameters. The last column indicates the function φ or the system which it satisfies.

Note that Corollary 4.9 cannot be applied in the situations above.

- **Classification of the Type (II) generalized Kasner space–times with constant scalar curvature**

  Let $M = I \times_{\varphi^{p_1}} F_1 \times_{\varphi^{p_2}} F_2$ be a Type (II) generalized Kasner space–time with constant scalar curvature. Then the parameters introduced before Proposition 4.11 satisfy $\zeta = p_1 + 2p_2$, $\eta = p_1^2 + 2p_2^2$ and

$$\tau = 2\zeta\frac{\varphi''}{\varphi} + [(\zeta-2)\zeta+\eta]\frac{(\varphi')^2}{\varphi^2} + \frac{\tau_{F_2}}{\varphi^{2p_2}}. \qquad \text{(csc-K-II.a)}$$

Note that τ_{F_2} must be constant if there exist a positive solution of (csc-K-II.a) (see also Proposition 3.5). We consider two principal cases with different subcases.

$\underline{\zeta = 0}$: if

$\underline{\eta = 0}$: then $p_1 = p_2 = 0$, $\tau = \tau_{F_2}$ and the corresponding metric is

$$-\mathrm{d}t^2 + g_{F_1} + g_{F_2}.$$

$\underline{\eta \neq 0}$: then $p_2 = -\frac{1}{2}p_1$ and $\eta = \frac{3}{2}p_1^2 = 6p_2^2$. The equation (csc-K-II.a) reduces to

$$\tau = \eta\frac{(\varphi')^2}{\varphi^2} + \frac{\tau_{F_2}}{\varphi^{2p_2}}. \qquad \text{(csc-K-II.b)}$$

$\underline{\zeta \neq 0}$: implies $\eta \neq 0$ and considering $0 < u = (\varphi^{\zeta})^{(1+\eta/\zeta^2)/2}$, Corollary 4.13 arises the relation

$$-\frac{4}{1+\frac{\eta}{\zeta^2}}u'' = -\tau u + \tau_{F_2}u^{1-4/(1+\eta/\zeta^2)(p_2/\zeta)}. \qquad \text{(csc-K-II.c)}$$

$\underline{\eta = \zeta^2}$: then $p_1 \neq 0$, either $p_2 = 0$ or $p_2 = -2p_1$, and $u = \varphi^{\zeta}$.

$\underline{\tau_{F_2} = 0}$: so the equation reduces to

$$-2u'' = -\tau u.$$

$\underline{\tau_{F_2} \neq 0}$: if

$\underline{p_2 = 0}$: the equation reduces to

$$-2u'' = -(\tau - \tau_{F_2})u.$$

$\underline{p_2 = -2p_1}$:

$$-2u'' = -\tau u + \tau_{F_2}u^{-1/3}.$$

$\underline{\eta \neq \zeta^2}$: then $p_2 \neq 0$ and $\frac{\eta}{\zeta^2} \geq \frac{1}{3}$.

$\underline{\tau_{F_2} = 0}$:

$$-\frac{4}{1+\frac{\eta}{\zeta^2}}u'' = -\tau u \qquad (5.1)$$

$\underline{\tau_{F_2} \neq 0}$:

$$-\frac{4}{1+\frac{\eta}{\zeta^2}}u'' = -\tau u + \tau_{F_2}u^{1-4/(1+\eta/\zeta^2)(p_2/\zeta)}. \qquad \text{(csc-K-II.c)}$$

Note that a particular subcase is $\frac{\eta}{\zeta^2} = \frac{1}{3}$. In fact, in this case, $p_1 = p_2 = \frac{\zeta}{3}$ (see Remark 4.2) and the latter equation reduces to the non-homogeneous linear ordinary differential equation

$$-3u'' = -\tau u + \tau_{F_2}.$$

Synthetically, remembering that in each case the corresponding metric may be written as $-\mathrm{d}t^2 + \varphi^{2p_1} g_{F_1} + \varphi^{2p_2} g_{F_2}$, we find that the only possibilities to have constant scalar curvature in a generalized Kasner space–time of type (II) are generated by where the conditions for τ must be imposed by the existence of positive solutions of the ordinary differential equations of the last column, on the corresponding interval I.

5.3. Type (III)

Let $M = I \times_{b_1} F_1 \times_{b_2} F_2 \times_{b_3} F_3$ be a type (III) warped product space–time with metric $g = -\mathrm{d}t^2 \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus b_3^2 g_{F_3}$. Then the scalar curvature τ of (M, g) is given as

$$\tau = 2\left(\frac{b_1''}{b_1} + \frac{b_2''}{b_2} + \frac{b_3''}{b_3} + \frac{b_1'b_2'}{b_1b_2} + \frac{b_2'b_3'}{b_2b_3} + \frac{b_1'b_3'}{b_1b_3}\right).$$

Note that $\tau_{F_i} = 0$, since $\dim(F_i) = 1$, for any $i \in \{1, 2, 3\}$.

If v_i is a vector field on F_i , for any $i \in \{1, 2, 3\}$ and $\bar{x} = \frac{\partial}{\partial t} + v_1 + v_2 + v_3$, then

$$\begin{aligned}\mathrm{Ric}(\bar{x}, \bar{x}) = {} & \left(b_1 b_1'' + \frac{b_1 b_1' b_2'}{b_2} + \frac{b_1 b_1' b_3'}{b_3}\right) g_{F_1}(v_1, v_1) \\ & + \left(b_2 b_2'' + \frac{b_2 b_2' b_1'}{b_1} + \frac{b_2 b_2' b_3'}{b_3}\right) g_{F_2}(v_2, v_2) \\ & + \left(b_3 b_3'' + \frac{b_3 b_3' b_1'}{b_1} + \frac{b_3 b_3' b_2'}{b_2}\right) g_{F_3}(v_3, v_3) - \frac{b_1''}{b_1} - \frac{b_2''}{b_2} - \frac{b_3''}{b_3}.\end{aligned}$$

Note that $\mathrm{Ric}_{F_i} \equiv 0$, since $\dim(F_i) = 1$, for any $i \in \{1, 2, 3\}$.

- **Classification of Einstein Type (III) generalized Kasner space–times**

  Let $M = I \times_{\varphi^{p_1}} F_1 \times_{\varphi^{p_2}} F_2 \times_{\varphi^{p_3}} F_3$ be an Einstein Type (III) generalized Kasner space–time. Then the parameters introduced before Proposition 4.3 satisfy $\zeta = p_1 + p_2 + p_3$, $\eta = p_1^2 + p_2^2 + p_3^2$. Hence the latter arises

$$\begin{cases} (\eta - \zeta)\frac{(\varphi')^2}{\varphi^2} + \zeta\frac{\varphi''}{\varphi} = \lambda, \\ p_1\left[(\zeta - 1)\frac{(\varphi')^2}{\varphi^2} + \frac{\varphi''}{\varphi}\right] = \lambda, \\ p_2\left[(\zeta - 1)\frac{(\varphi')^2}{\varphi^2} + \frac{\varphi''}{\varphi}\right] = \lambda, \\ p_3\left[(\zeta - 1)\frac{(\varphi')^2}{\varphi^2} + \frac{\varphi''}{\varphi}\right] = \lambda. \end{cases} \qquad \text{(E-K-III)}$$

Note that adding the last three equations, there results

$$\zeta\left[(\zeta-1)\frac{(\varphi')^2}{\varphi^2}+\frac{\varphi''}{\varphi}\right]=3\lambda. \tag{5.2}$$

Consider now two cases, namely

$\underline{\zeta=0}$: then applying (5.2), we obtain $\lambda=0$.

$\underline{\eta=0}$: thus $p_i=0$ for all i. Hence the corresponding metric is

$$-\mathrm{d}t^2+g_{F_1}+g_{F_2}+g_{F_3}.$$

$\underline{\eta\neq 0}$: the system reduces to

$$\begin{cases}\eta\frac{(\varphi')^2}{\varphi^2}=0,\\ p_i\left[-\frac{(\varphi')^2}{\varphi^2}+\frac{\varphi''}{\varphi}\right]=0 \text{ for all } i=1,2,3\end{cases} \tag{E-K-IIIi }$$

then φ is constant φ_0 . Thus the corresponding metric is

$$-\mathrm{d}t^2+\varphi_0^{2p_1}g_{F_1}+\varphi_0^{2p_2}g_{F_2}+\varphi_0^{2p_3}g_{F_3}.$$

$\underline{\zeta\neq 0}$: thus $\eta\neq 0$ and by Remark 4.4 the system reduces to

$$\begin{aligned}&\frac{\zeta^2}{\eta}\frac{(\varphi^{\zeta(\eta/\zeta^2)})''}{\varphi^{\zeta(\eta/\zeta^2)}}=\lambda,\\ &\frac{p_i}{\zeta}\frac{(\varphi^{\zeta})''}{\varphi^{\zeta}}=\lambda \quad \text{for all } i=1,2,3.\end{aligned} \tag{E-K-IIIii }$$

Adding the last three equations in (E-K-IIIii), we obtain that

$$\frac{(\varphi^{\zeta})''}{\varphi^{\zeta}}=3\lambda. \tag{5.3}$$

$\underline{\eta=\zeta^2}$: then (5.2) and (5.3), give $\lambda=0$. Thus, the corresponding metric is

$$-\mathrm{d}t^2+\varphi^{2p_1}g_{F_1}+\varphi^{2p_2}g_{F_2}+\varphi^{2p_3}g_{F_3},$$

where φ satisfies $(\varphi^{\zeta};0)$.

$\underline{\eta\neq\zeta^2}$: then at least two p_i 's are $\neq 0$. So if

$\underline{\lambda=0}$: then the first equation implies $\varphi^{\zeta}=(\mathcal{A}t+\mathcal{B})^{\zeta^2/\eta}$ and $(\varphi^{\zeta})''=\frac{\zeta^2}{\eta}\left(\frac{\zeta^2}{\eta}-1\right)(\mathcal{A}t+\mathcal{B})^{\zeta^2/\eta-2}\mathcal{A}^2$. Then by (5.3) results $\mathcal{A}=0$, so φ^{ζ} is constant and φ is a positive constant φ_0 . Thus the corresponding metric is

$$-\mathrm{d}t^2+\varphi_0^{2p_1}g_{F_1}+\varphi_0^{2p_2}g_{F_2}+\varphi_0^{2p_3}g_{F_3}.$$

$\underline{\lambda\neq 0}$: then all p_i 's are $\neq 0$ and all of them are equals, so that $p_1=p_2=p_3=\frac{\zeta}{3}$. So $\eta=\frac{\zeta^2}{3}$ and $\frac{\eta}{\zeta^2}=\frac{1}{3}$. Thus the system reduces to

$$3\frac{(\varphi^{\zeta(1/3)})''}{\varphi^{\zeta(1/3)}}=\frac{1}{3}\frac{(\varphi^{\zeta})''}{\varphi^{\zeta}}=\lambda,$$

which is equivalent to the solvable system $(\varphi^{\zeta};3\lambda;*)$. Note that λ must be >0 .

Table 3 specifies the only possible Einstein generalized Kasner space–times of type (III) with the corresponding parameters. Like for the table of Type (II), the last column indicates the function φ or the system which it satisfies.

This example may be easily generalized to the situation all the F_i 's are Ricci flat, considering $S = \sum_{i=1}^{m} s_i > 1$ instead of 3.

- **Classification of Type (III) generalized Kasner space–times with constant scalar curvature**

  Let $M = I \times_{\varphi^{p_1}} F_1 \times_{\varphi^{p_2}} F_2 \times_{\varphi^{p_3}} F_3$ be a Type (III) generalized Kasner manifold with constant scalar curvature. Then the parameters introduced before Proposition 4.11 satisfy $\zeta = p_1 + p_2 + p_3$, $\eta = p_1^2 + p_2^2 + p_3^2$. Thus, this case is already included in the analysis of Example 4.15.

We will close this section by an example and the following comment which gives some preliminary ideas about our future plans on this topic (see also the last section for details).

Example 5.2. Let $M = I \times_{\varphi^{p_1}} \mathbb{S}_3 \times_{\varphi^{p_2}} \mathbb{S}_2$ be a generalized Kasner manifold with constant scalar curvature. Then the parameters introduced before Proposition 4.11 are given by $\zeta = 3p_1 + 2p_2$, $\eta = 3p_1^2 + 2p_2^2$. Consider now $p_1 = 1$ and $p_2 = -1$, then $\zeta = 1$ and $\eta = 5$. Hence, applying Corollary 4.13 the latter conditions arise for $u = \varphi^3$ the problem

$$-\tfrac{2}{3}u'' + \tau u = \tau_{\mathbb{S}_3} u^{1-2/3} + \tau_{\mathbb{S}_2} u^{1+2/3}, \quad u > 0, \tag{5.4}$$

where $\tau_{\mathbb{S}_3}, \tau_{\mathbb{S}_2} > 0$ are the constant scalar curvatures of the corresponding spheres. Note that the equation in (5.4) has always the constant solution zero and there exists $\tau_1 > 0$ such that for $\tau = \tau_1$ there is only one constant solution of (5.4) and for any $\tau > \tau_1$ there are two constant solutions of (5.4), so that there exists a range of τ 's, $(\tau_1, +\infty)$, where the problem (5.4) has multiplicity of solutions; while there is no constant solutions when $\tau < \tau_1$.

On the other hand, as in Example 5.2, considering $\mathbb{S}_3$ instead of $\mathbb{S}_2$ with the same values of p_1 and p_2 , i.e., $M = I \times_{\varphi^{p_1}} \mathbb{S}_3 \times_{\varphi^{p_2}} \mathbb{S}_3$, results $\zeta = 3p_1 + 3p_2 = 0$, $\eta = 3p_1^2 + 3p_2^2 = 6$. Hence, applying Proposition 4.11 the latter conditions arise the problem

$$-6(\varphi')^2 = -\tau\varphi^2 + \tau_{\mathbb{S}_3} + \tau_{\mathbb{S}_3}\varphi^4, \quad \varphi > 0. \tag{5.5}$$

The equation in (5.5) does not have the constant solution zero. Furthermore there is no constant solution of (5.5) if $\tau < 2\tau_{\mathbb{S}_3}$, there is only one constant solution of (5.5) if $\tau = 2\tau_{\mathbb{S}_3}$ and two constant solutions of (5.5) if $\tau > 2\tau_{\mathbb{S}_3}$.

The cases considered above are just some examples for the different types of differential equations involved in the problem of constant scalar curvature when the dimensions, curvatures and parameters have different values. In a future article, we deal with the problem of constant scalar curvature of a pseudo-Riemannian generalized Kasner manifolds with a base of dimension greater than or equal to 1. This problem carries to nonlinear partial differential equations with concave–convex nonlinearities like in (5.4), among others. Nonlinear elliptic problems with such nonlinearities have been extensively studied in bounded domains of $\mathbb{R}^n$, after the central article of Ambrosetti et al. [1], in which the authors studied the problem of multiplicity of solutions under Dirichlet conditions. The problem of constant scalar curvature in a generalized Kasner manifolds with base of dimension greater than or equal to 1 is one of the first examples where those nonlinearities appear naturally. Another related case is the base conformal warped products, studied in [26].

6. BTZ (2 + 1)-black hole solutions

1Now we consider BTZ (2 + 1)-black hole solutions and give another characterization of (BTZ) black hole solutions mentioned in Section 2 (for further details see [8,9,42,59]) in order to apply the results obtained in this paper.

All the cases considered in [42], can be obtained applying the *formal* approach that follows. By considering the corresponding square lapse function N^2 , the related three-dimensional, (2 + 1) -space–time model can be expressed as a (2 + 1) multiply generalized Robertson–Walker space–time, i.e.,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + b_1^2(t)\,\mathrm{d}x^2 + b_2^2(t)\,\mathrm{d}\phi^2, \tag{6.1}$$

where

$$b_1(t) = N(F^{-1}(t)), \qquad b_2(t) = F^{-1}(t) \tag{6.2}$$

with

$$F(r) = \int_a^r \frac{1}{N(\mu)}\,\mathrm{d}\mu \tag{6.3}$$

and F^{-1} the inverse function of F (assuming that there exists) and a is an appropriate constant that is most of the time related to the event horizon.

Recalling

$$1 = \frac{\mathrm{d}}{\mathrm{d}t}(F \circ F^{-1})(t) = F'(F^{-1}(t))(F^{-1})'(t), \tag{6.4}$$

we obtain the following properties by applying the chain rule. Here, note that all the functions depend on the variable t and the derivatives are taken with respect to the corresponding arguments.

- $b_1 = N(b_2)$.
- $b_2' = N(F^{-1}) = b_1$.
- $b_2' = N(b_2)$.
- $b_1' = b_2'' = N'(b_2)b_2' = N'(b_2)b_1$.
- $b_1'' = N''(b_2)b_2'b_1 + N'(b_2)b_1' = N''(b_2)b_1^2 + (N'(b_2))^2 b_1$.

Thus,

$$\begin{aligned}
&\bullet\ \frac{b_1''}{b_1} = N''(b_2)b_1 + (N'(b_2))^2 = N''(b_2)N(b_2) + (N'(b_2))^2 \\
&\qquad = (N'N)'(b_2) = \tfrac{1}{2}(N^2)''(b_2). \\
&\bullet\ \frac{b_2''}{b_2} = N'(b_2)\frac{N(b_2)}{b_2} = \frac{1}{2}\frac{(N^2)'(b_2)}{b_2}. \\
&\bullet\ \frac{b_1'}{b_1}\frac{b_2'}{b_2} = \frac{b_2''}{b_2}.
\end{aligned} \tag{6.5}$$

On the other hand, by Corollary 2.7 applied to the metric (6.1), with $s_1 = s_2 = 1$. The scalar curvature of the corresponding space–time is given by

$$\tau = 2\left(\frac{b_1''}{b_1} + \frac{b_2''}{b_2} + \frac{b_1'}{b_1}\frac{b_2'}{b_2}\right) = 2\left(\frac{b_1''}{b_1} + 2\frac{b_2''}{b_2}\right)$$
$$= 2\left(N''(b_2)N(b_2) + (N'(b_2))^2 + 2N'(b_2)\frac{N(b_2)}{b_2}\right) = (N^2)''(b_2) + 2\frac{(N^2)'(b_2)}{b_2}. \tag{6.6}$$

Note that, the latter is an expression of the scalar curvature as an operator in the square lapse function. Remember that $b_2 = F^{-1}$.

About the Ricci tensor, applying our Proposition 3.2 and Theorem 3.3 and by considering again $s_1 = s_2 = 1$, Theorem 3.3 says that the metric (6.1) is Einstein with λ if and only if

$$\begin{aligned} &\frac{b_1''}{b_1} + \frac{b_2''}{b_2} = \lambda, \\ &\frac{b_1''}{b_1} + \frac{b_1'}{b_1}\frac{b_2'}{b_2} = \lambda, \\ &\frac{b_2''}{b_2} + \frac{b_2'}{b_2}\frac{b_1'}{b_1} = \lambda. \end{aligned} \tag{6.7}$$

On the other hand by making use of (6.5), the system (6.7) is equivalent to (all the functions are evaluated in $r = b_2$)

$$\begin{aligned} &(N^2)'' + \frac{(N^2)'}{r} = 2\lambda, \\ &(N^2)'' + \frac{(N^2)'}{r} = 2\lambda, \\ &\frac{(N^2)'}{r} = \lambda, \end{aligned} \tag{6.8}$$

or moreover to the following

$$\begin{aligned} &(N^2)'' + \frac{(N^2)'}{r} = 2\lambda, \\ &\frac{(N^2)'}{r} = \lambda. \end{aligned} \tag{6.9}$$

Thus, we have

$$(N^2)'' = \lambda. \tag{6.10}$$

Hence:

$$N^2(r) = \frac{\lambda}{2}r^2 + c_1 r + c_2 \tag{6.11}$$

with c_1 and c_2 suitable constants. But, since $(N^2)'(r) = \lambda r + c_1$, the second equation of (6.9) is verified if and only if $c_1 = 0$. So, we have proved the following results.

Proposition 6.1. *Suppose that we have a* $(2+1)$ *-Lorentzian multiply warped product with the metric given by* (6.1)*, where* b_1 *and* b_2 *satisfying both* (6.2) *and* (6.3)*. The space–time is Einstein with Ricci curvature* λ *if and only if the square lapse function* N^2 *satisfies* (6.11)*, with* $c_1 = 0$ *and a suitable constant* c_2 *.*

Notice that the static (BTZ) and the static (dS) black hole solutions considered in [42] satisfy Proposition 6.1. Thus they are Einstein multiply warped product space–times.

Remark 6.2. Remark that if N^2 satisfies (6.11) with $c_1 = 0$, then an application of (6.6) gives the constancy of the scalar curvature $\tau = 3\lambda$, as desired. Note that this result agrees with the ones obtained in [42].

Furthermore, the following just follows from the solution of the involved second order linear ordinary differential equation arisen by the expression (6.6).

Proposition 6.3. *Suppose that there is a* $(2+1)$ *-Lorentzian multiply warped product with the metric given by* (6.1)*, where* b_1 *and* b_2 *verifying* (6.2) *and* (6.3)*. The space–time has constant scalar curvature* $\tau = \lambda$ *if and only if the square lapse function* N^2 *has the form*

$$N^2(r) = -c_1 \frac{1}{r} + \frac{\lambda}{6} r^2 + c_2, \tag{6.12}$$

with suitable constants c_1 *and* c_2 *.*

Note that Proposition 6.3 agrees with Remark 6.2.

7. Conclusions

Now, we would like to summarize the content of the paper and to make some concluding remarks. In a brief, we studied expressions that relate the Ricci (respectively, scalar) curvature of a multiply warped product with the Ricci (respectively, scalar) curvatures of its base and fibers as well as warping functions.

By using expressions obtained in the paper, we proved necessary and sufficient conditions for a multiply generalized Robertson–Walker space–time to be Einstein or to have constant scalar curvature.

Furthermore, we introduced and considered a kind of generalization of Kasner space–times, which is closely related to recent applications in cosmology where metrics of the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \sum_{i=1}^{k} \mathrm{e}^{2\alpha_i}\,\mathrm{d}x_i^2 \quad \text{with } \alpha_i = \alpha_i(t) \tag{7.1}$$

are frequently considered (see [39,69]; for other recent topics concerned Kasner type metrics see for instance [23,36,44,45,58,63,73,74]). If each warping function $\mathrm{e}^{2\alpha_i}$ is expressed as

$$\mathrm{e}^{2\alpha_i} = \varphi_i^{2p_i} \quad \text{with } \varphi_i = \mathrm{e}^{\alpha_i/p_i} \tag{7.2}$$

for suitable p_i 's, then (7.1) takes the form

$$ds^2 = -dt^2 + \sum_{i=1}^{k} \varphi_i^{2p_i} dx_i^2. \tag{7.3}$$

Our generalization of Kasner space–times corresponds exactly to the case in which the φ_i 's are independent of i. More explicitly, $\alpha_i = p_i\alpha$ in Eq. (7.2), with $\alpha = \alpha(t)$ for a sufficiently regular fixed function. Note that a classical Kasner space–time corresponds to the case of $\alpha \equiv 1$ (see [55] also).

By applying Lemma 2.8, we obtained useful expressions for the Ricci tensor and the scalar curvature of generalized Robertson–Walker and generalized Kasner space–times. These expressions allowed us to classify possible Einstein (respectively, with constant scalar curvature) generalized Kasner space–times of dimension 4. We also obtained some partial results for greater dimensions.

Finally, in order to study curvature properties of multiply warped product space–times associated to the BTZ $(2 + 1)$ -dimensional black hole solutions, we made applications of the previously obtained curvature formulas. As a consequence, we characterized the Einstein BTZ (respectively, with constant scalar curvature), in terms of the square lapse function.

In forthcoming papers we plan to focus on a specific generalization of the structures studied here, which is particularly useful in different fields such as relativity, extra-dimension theories (Kaluza-Klein, Randall–Sundrum), string and super-gravity theories, spectrum of Laplace–Beltrami operators on p-forms, among others. Roughly speaking, we will consider a mixed structure between a multiply warped product and a conformal change in the base. Naturally, our main interest is the study of curvature properties. As we have made progress on this subject, we realized that these curvature related properties are interesting and worth to study not only for the physical point of view (see for instance, the several recent works of Gauntlett, Maldacena, Argurio, Schmidt, among many others), but also for exclusive nonlinear partial differential equations involved. Indeed, the curvature related questions arise problems of existence, uniqueness, bifurcation, study of critical points, etc. (see Example 5.2 above and the different works of Aubin, Hebey, Yau, Ambrosetti, Choquet-Bruat among others).

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References

[1] A. Ambrosetti, N. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, J. Funct. Anal. 122 (1994) 519–543.

[2] L.J. Alías, A. Romero, M. Sánchez, Spacelike hypersurfaces of constant mean curvature and Calabi–Bernstein type problems, Tohôku Math. J. 49 (1997) 337–345.

[3] D.E. Allison, Energy conditions in standard static space–times, Gen. Relativity Gravitation 20 (2) (1988) 115–122.

[4] D.E. Allison, Geodesic completeness in static spacetimes, Geom. Dedicata 26 (1988) 85–97.

[5] D.E. Allison, B. Ünal, Geodesic structure of standard static space–times, J. Geom. Phys. 46 (2) (2003) 193–200.

[6] T. Aubin, Some Nonlinear Problems in Riemannian Geometry, Springer-Verlag, New York, 1998.

[7] M. Badiale, F. Dobarro, Some existence results for sublinear elliptic problems in R^N , Funkcial. Ekvac. 39 (2) (1996) 183–202.

[8] M. Bañados, C. Henneaux, C. Teitelboim, J. Zanelli, Geometry of $2 + 1$ black hole, Phys. Rev. D 48 (1993) 1506–1525.

[9] M. Bañados, C. Teitelboim, J. Zanelli, The black hole in three-dimensional spacetime, Phys. Rev. Lett. 69 (1992) 1849–1851.

[10] J.K. Beem, Lorentzian Geometry in the Large, Mathematics of Gravitation, Part I, Lorentzian Geometry and Einstein Equations, vol. 41, Banach Center Publications, 1997, pp. 11–20.

[11] J.K. Beem, Stability of geodesic structures, Proceedings of the Second World Congress of Nonlinear Analysts, Part 1 (Athens 1996) Nonlinear Anal., vol. 30, 1997, pp. 567–570.

[12] J.K. Beem, P. Ehrlich, Global Lorentzian Geometry, 1st ed., Markel-Deccer, New York, 1981.

[13] J.K. Beem, P. Ehrlich, K. Easley, Global Lorentzian Geometry, 2nd ed., Markel-Deccer, New York, 1996.

[14] J.K. Beem, P. Ehrlich, T.G. Powell, in: T.M. Rassias, G.M. Rassias (Eds.), Warped Product Manifolds in Relativity in Selected Studies: A Volume Dedicated to the Memory of Albert Einstein, North-Holland, Armsterdarm, 1982, pp. 41–56.

[15] J.K. Beem, P. Parker, Pseudoconvexity and geodesic connectedness, Ann. Mat. Pura Appl. 155 (1989) 137–142.

[16] A.L. Besse, Einstein Manifolds, Springer-Verlag, Germany, 1987.

[17] R.L. Bishop, B. O'Neill, Manifolds of negative curvature, Trans. Am. Math. Soc. 145 (1969) 1–49.

[18] J. Chabrowski, J.M. Bezzera do Ó, On semilinear elliptic equations involving concave and convex nonlinearities, Math. Nachr. 233/234 (2002) 55–76.

[19] J. Choi, Multiply warped products with nonsmooth metrics, J. Math. Phys. 41 (2000) 8163–8169.

[20] J. Choi, S.-T. Hong, Warped product approach to universe with nonsmooth scale factor, J. Math. Phys. 45 (2004) 642–651.

[21] B.Y. Chen, F. Dillen, L. Verstraelen, L. Vranken, Lagrangian isometric immersions of a real-space-form $M^n(c)$ into a complex-space-form $\tilde{M}^n(4c)$, Math. Proc. Camb. Philos. Soc. 124 (1998) 107–125.

[22] V. Coti Zelati, F. Dobarro, R. Musina, Prescribing scalar curvature in warped products, Ricerche Mat. 46 (1) (1997) 61–76.

[23] M.P. Dąbrowski, Kasner Asymptotics of Mixmaster Hořava-Witten Cosmology. arXiv:hep-th/9911217.

[24] F. Dobarro, E. Lami Dozo, Scalar curvatures and warped products of Riemannian geometry, Trans. Am. Math. Soc. 303 (1987) 161–168.

[25] F. Dobarro, B. Ünal, Special standard static space times, Nonlin. Anal. TMA 59 (5) (2004) 759–770 [arXiv:math.FA/0406054].

[26] F. Dobarro, B. Ünal, Curvature in special base conformal warped products (2004) [arXiv:math.DG/0412436].

[27] K.L. Duggal, Constant scalar curvature and warped product globally null manifolds, J. Geom. Phys. 43 (2002) 327–340.

[28] K.L. Easley, Local existence of warped product metrics, Ph.D. Thesis, University of Missouri-Columbia, 1991.

[29] P.E. Ehrlich, Y.-T. Jung, S.-B. Kim, Constant scalar curvatures on warped product manifolds, Tsukuba J. Math. 20 (1) (1996) 239–256.

[30] P.E. Ehrlich, Y.-T. Jung, S.-B. Kim, C.-G. Shin, Partial differential equations and scalar curvature of warped product manifolds, Nonlin. Anal. 44 (4) (2001) 545–553.

[31] P. Ehrlich, M. Sánchez, Some semi-Riemannian volume comparison theorems, Tohôku Math. J. 52 (2000) 285–314.

[32] M. Fernández-López, E. García-Río, D.N. Kupeli, B. Ünal, A curvature condition for a doubly twisted product to be a warped product, Manuscripta Math. 106 (2) (2001) 213–217.
[33] J. Flores, M. Sánchez, Geodesic connectedness and conjugate points in GRW space–times, J. Geom. Phys. 36 (3/4) (2000) 285–314.
[34] J. Flores, M. Sánchez, Geodesic connectedness of multiwarped space–times, J. Diff. Eqs. 186 (1) (2002) 1–30.
[35] G.B. Folland, Real Analysis, Modern Techniques and Their Applications, 2nd ed., Wiley, U.S.A, 1999.
[36] A.V. Frolov, Kasner-AdS spacetime and anisotropic brane-world cosmology. arXiv:gr-qc/0102064.
[37] G. Ganchev, V. Mihoa, Riemannian submanifolds of quasi-constant sectional curvatures, J. Reine Angew. Math. 522 (2000) 119–141.
[38] A. Gebarowski, On Einstein warped products, Tensor (N. S.) 52 (3) (1993) 204–207.
[39] P. Halpern, The Behavior of Kasner Cosmologies with Induced Matter. arXiv:gr-qc/0010090.
[40] A. Harvey, Geodesics in Kasner universes, Phys. Rev. D 39 (1989) 673–676.
[41] S.W. Hawking, G.F.R. Ellis, The Large Scale Structure of Space–Time, Cambridge University Press, UK, 1973.
[42] S.-T. Hong, J. Choi, Y.-J. Park, $(2+1)$ BTZ black hole and multiply warped product spacetimes, Gen. Relativity Gravitation 35 (2003) 2105–2116.
[43] M. Ito, Warped geometry in higher dimensions with an orbitfold extra dimension. arXiv:hep-th/0105186.
[44] V.D. Ivashchuk, D. Singleton, Composite electric S-brane solutions with maximal number of branes. arXiv:hep-th/0407224.
[45] L. Järv, T. Mohaupt, F. Saueressig, M-theory cosmologies from Sigular Calabi–Yau compactifications. arXiv:hep-th/0310174.
[46] E. Kasner, Geometrical theorems in Einstein's cosmological equations, Am. J. Math. 43 (1921) 217.
[47] M.O. Katanaev, T. Klösch, W. Kummer, Global properties of warped solutions in general relativity, Ann. Phys. 276 (2) (1999) 191–222.
[48] J.L. Kazdan, Prescribing the curvature of a Riemannian manifold, in: American Mathematical Society CBMS Regional Conference Series in Mathematics, vol. 57, Providence, RI, 1985.
[49] D.-S. Kim, Einstein warped product spaces, Honam Math. J. 22 (1) (2000) 107–111.
[50] D.-S. Kim, Y.H. Kim, Compact Einstein warped product spaces with nonpositive scalar curvature, Proc. Am. Math. Soc. 131 (8) (2003) 2573–2576.
[51] J.-S. Kim, H.-S. Chung, On Einstein warped products with compact Riemannian base, Honam Math. J. 20 (1) (1998) 153–159.
[52] O. Kobayashi, M. Obata, Certain mathematical problems on static models in general relativity, in: S.S. Chern, W. Wen-tsün (Eds.), Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, vol. 3, 1980, pp. 1333–1344.
[53] O. Kobayashi, M. Obata, Conformally-flatness and static space–time, Manifolds and Lie Groups (Notre Dame, Ind., 1980), Progr. Math., vol. 14, Birkhäuser, Boston, MA, 1981, pp. 197–206.
[54] N. Koike, The decomposition of curvature netted hypersurface, Geom. Dedicata 54 (1995) 1–11.
[55] S.S. Kokarev, Multidimensional generalization of Kasner solution. arXiv:gr-gc/9510059.
[56] D. Kramer, H. Stephani, E. Herlt, M. MacCallum, Exact Solutions of Einstein's Field Equations, Cambridge University Press, UK, 1980.
[57] M. Lohnherr, H. Reckziegel, On ruled real hypersurfaces in complex space forms, Geom. Dedicata 74 (1999) 267–286.
[58] M. Maceda, On the Wheeler–DeWitt equation for Kasner-like cosmologies. arXiv:hep-th/0408239.
[59] C. Matínez, C. Teitelboim, J. Zanelli, Charged rotating balck hole in three spacetime dimensions, Phys. Rev. D 61 (2000) 1–8.
[60] M. Mikkelsen, Some results on standard static space–times with perfect fluid, Diff. Geom. Appl. 1 (1992) 211–220.
[61] C.W. Misner, J.A. Wheeler, K.S. Thorne, Gravitation, W.H. Freeman, San Francisco, 1973.
[62] B. O'Neill, Semi-Riemannian Geometry, Academic Press, NY, 1983.
[63] G. Papadopoulos, Power-law singularities in string theory and M-theory. arXiv:hep-th/0404172.
[64] T.G. Powell, Lorentzian manifolds with non-smooth metrics and warped products, Ph.D. Thesis, University of Missouri-Columbia, 1982.

[65] A. Romero, M. Sánchez, On the completeness of certain families of semi-Riemannian manifolds, Geom. Dedicata 53 (1994) 103–117.

[66] M. Sánchez, Geodesic connectedness of in generalized Reissner–Nordström type Lorentz manifolds, Gen. Relativity Gravitation 29 (1997) 1023–1037.

[67] M. Sánchez, On the geometry of generalized Robertson–Walker spacetimes: geodesics, Gen. Relativity Gravitation 30 (1998) 915–932.

[68] M. Sánchez, On the Geometry of Generalized Robertson–Walker spacetimes: curvature and Killing fields, J. Geom. Phys. 31 (1999) 1–15.

[69] J. Socorro, V.M. Villanueva, L.O. Pimentel, Classical solutions in five-dimensional induced matter theory and its relations with an imperfect fluid, Int. J. Mod. Phys. A 11 (1996) 5495 arXiv:gr-qc/9605012.

[70] B. Ünal, Doubly warped products, Ph.D. Thesis, University of Missouri-Columbia, 2000.

[71] B. Ünal, Doubly warped products, Differential Geom. Appl. 15 (3) (2001) 253–263.

[72] B. Ünal, Multiply warped products, J. Geom. Phys. 34 (3/4) (2000) 287–301.

[73] R.J. Van Den Hoogen, A. Horne, Brane-world Cosmologies with non-local bulk effects. arXiv:gr-qc/0408014.

[74] Yu.S. Vladimirov, S.S. Kokarev, 4D homogeneous isotropic cosmological models generated by the 5D vacuum. arXiv:gr-qc/0210067.

[75] X. Zhang, The positive mass theorem in general relativity, AMS (Am. Math. Soc.)/IP Stud. Adv. Math. 29 (2001) 227–233.